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THE CASE OF THE FORGETFUL BURGLAR

CAXTON FOSTER AND ANATOL RAPOPORT, Mental Health Research Institute, University of Michigan

While there has been considerable attention paid to random walk problems in one or more dimensions, there has as yet been little investigation of walks in which each previously occupied point becomes an absorbing site, which, if occupied again, ends the walk. Such a walk is a model for the case of the forgetful burglar who, working one side of a street, feels compelled to pass no more than one house by without plying his trade. Thus coming out of a particular house he may enter the one next door or skip that and enter the succeeding one. He becomes so absorbed in his burgling that he forgets from which direction he entered a particular house and which houses on the street he has already entered. If he is so unfortunate as to re-enter a house, the already aroused occupants detain him to his permanent disadvantage. The question of interest is "How long, on the average, can he expect to keep this up?" And if many such burglars operate in this fashion, "What is the distribution of the lengths of their paths?"

This problem is of course generalizable to burglars operating in two, three, or even more dimensions and skipping none, one, two, or more houses with either random distribution of choices or with some sort of bias in the choice. One could also include an arbitrary "decay" time for the arousal of the home owners. The general solution of this type of problem would be of interest not only to forgetful burglars but also to itinerant medicine shows, standard random-type drunkards with superstitions against retracing steps, some neutrons in chain reactions, impulses in a neural network, and to sociologists studying sociograms. Indeed, it was from an attempt to build a model for sociograms that interest in this problem originally arose.

The special case of the one-dimensional walk with only single steps is trivial, because the first reversal of direction terminates the walk. The authors have not succeeded in finding a general solution for problems of this type. But the special case of the one-dimensional walk in which steps may be either singles or doubles has been solved and may have heuristic value for a future generalization.

Consider, then, an infinite one-dimensional grid of sites. Starting from an arbitrary site, the walker takes with equal probability one or two steps in either direction. When a site previously occupied is re-entered, the walk terminates. It is required to calculate the frequency distribution of walk lengths.

1. Definitions. A *chain* is a path that has not yet terminated, that is, one in which no site has been entered more than once.

A *closed hook* is a chain which starts to the left (right) and ends at the site immediately to the right (left) of the starting site. We will call a closed hook which starts to the left, a left closed hook and one that starts to the right, a right

closed hook. We will call the starting site "0," the site immediately to the right "+1," the site immediately to the left as "−1." We will sometimes abbreviate "closed hook" to "hook."

A *trip* is a chain without reversal of direction. Obviously any chain is composed of trips of alternating directions.

A *coil* is a chain which proceeds steadily in one direction except for possible retrograde single steps, and ends on an extreme site of the chain.

An *open hook* is a chain which starts to the right (left) and ends at +1 (−1), and has at least one reversal of direction.

Note that an additional double step in the same direction changes an open hook into a closed hook. Right and left open hooks are analogous to right and left closed hooks, if the starting and ending sites are interchanged.

Examples of a closed hook, a coil, and an open hook are shown in Figure 1. In the proofs that follow it will be assumed that each path starts to the left. Complementary lemmas follow by symmetry.

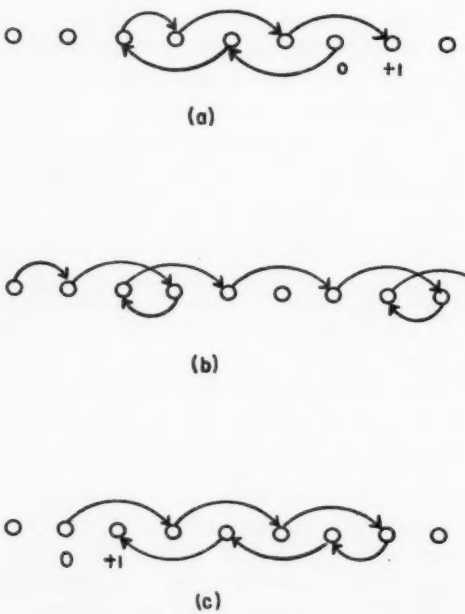

FIG. 1. (a) closed hook; (b) coil; (c) open hook.

LEMMA 1. *A two-step left closed hook starts with a single step. Conversely, if a closed hook starts with a single step, it is a two-step closed hook.*

Proof. If the first step is a double to the left, the hook cannot be completed

on the second step, because a single to the right lands on -1 and a double to the right terminates the path. Conversely if the first step is a single, a reversal must follow immediately; otherwise two sites $(0, -1)$ will have to be skipped to complete the hook, which is impossible. The reversal must be a double step, since otherwise the path is terminated. But the double step completes the hook on the second step, hence the lemma.

LEMMA 2. *In a closed hook exactly one reversal of direction occurs.*

Proof. At least one reversal is obviously necessary. We must show that no more than one can occur. If the trip to the right following the first reversal carries the path beyond 0, then $+1$ must be occupied, otherwise two consecutive sites $(0, +1)$ must be skipped, which is impossible. But then either $+1$ ends the hook and the lemma is proved, or $+1$ must be occupied again, a contradiction, since a hook is not a terminated path. If the first trip to the right does not carry past 0, then $+1$ must be reached after a subsequent trip to the right. But this is impossible since a double reversal following a trip to the left establishes a barrier of two consecutive occupied sites which no subsequent trip to the right can cross.

LEMMA 3. *All the steps of a left hook of h steps are completely determined.*

Proof. By Lemma 1, this is true for $h=2$. If $h>2$, the first step must be double by Lemma 1, and so must every subsequent step to the left except possibly the last one; otherwise some consecutive pair of sites will have to be skipped on the trip to the right, which is impossible. If the last (kth) step to the left is double, the first step to the right must be single followed by k double steps, making $h=2k+1$ in all. If the kth step to the left is single, the first step to the right must be double, followed by $k-1$ double steps, making $h=2k$ in all. Thus all steps are determined for $h=2k+1$ (odd) and for $h=2k$ (even). But this exhausts all cases.

COROLLARY 1. *At the completion of a left closed hook of h steps, h consecutive sites to the left of $+1$ are occupied.*

COROLLARY 2. *After completion of a left closed hook, the chain can continue only to the right.*

LEMMA 4. *If two consecutive reversals occur in a chain (that is reversals separating three consecutive trips), the trip following the first of these reversals either carries beyond $+1$, or consists of one single step.*

Proof. Suppose, on the contrary, that the first of these reversals is followed by more than one step but does not carry the trip beyond $+1$. Then at the end of this trip there are more than one occupied sites to the left, and the second reversal is impossible.

LEMMA 5. *All the steps of an open hook are determined.*

Proof. An open hook is equivalent to a closed hook if the starting and ending sites are interchanged.

COROLLARY. *An open hook has exactly one reversal.*

These results can be summarized in the principal theorem on the structure of chains.

THEOREM. *Every chain is composed of a closed hook of h steps, a coil of c steps, and an open hook of y steps.* ($h, c, y \geqq 0$, $h \neq 1$, $y \neq 1$).

Proof. Suppose the first h steps complete a left hook. Then by Corollary 2 the chain must continue to the right. If reversals occur, then the first and second reversals must be of the type occurring in a coil; so also will be the third and fourth, the fifth and sixth, *etc.*, by Lemma 4.

Thus if the last reversal is to the right, we have a closed hook followed by a coil. If the last reversal is to the left, the number of steps in the trip to the left cannot be specified. But then each step completes some open hook since it must enter a site immediately to the right of another, which can be taken to be the origin of a right open hook. Then we have a closed hook, a coil, and an open hook. If $+1$ is not reached following the first reversal, we have a coil to the left and a left open hook or a left open hook alone. This exhausts all possibilities.

2. It remains to calculate the probability of occurrence of a chain consisting of a closed hook of h steps, a coil of c steps, and an open hook of y steps. We shall do so for the case where the chain starts with a left hook. The resulting probabilities will then be doubled to get the final result.

Since by Lemma 3, all the steps of a left hook are determined, we have for the probability of occurrence of a left hook of h steps

$$P_h(h) = (1/4)^h. \tag{1}$$

Let the coil which follows the hook have c steps in all and ρ double reversals. Then 3ρ steps are specified completely, since a double reversal, given a direction, consists of a double, a single, and a double in specified directions. The remaining $c-3\rho$ steps are specified as to direction only, their sizes being arbitrary. These ρ reverses can be thought of as determining ρ partitions and hence $\rho+1$ bins, into which the remaining $c-3\rho$ steps may be placed in any way whatsoever. Therefore there are

$$\frac{[(c-3\rho)+(\rho+1)-1]!}{(c-3\rho)![(\rho+1)-1]!} = \frac{(c-2\rho)!}{(c-3\rho)!\rho!} \tag{2}$$

ways of making a coil with c steps and ρ double reversals. The probability of any one of these is

$$P_1(c, \rho) = (1/4)^{3\rho} \cdot (1/2)^{c-3\rho} = (1/2)^{c+3\rho}. \tag{3}$$

Since ρ may range from 0 to $[c/3]$, the probability of making a coil of c steps with any number of reversals is

$$P_c(c) = \sum_{\rho=0}^{[c/3]} (1/2)^{c+3\rho} \frac{(c-2\rho)!}{(c-3\rho)!\rho!}. \tag{4}$$

If a hook is followed by a coil containing x steps between them, then the probability of making a "hooked coil" of x steps is

$$P_{hc}(x) = \sum_{h=2}^{x} P_c(x-h)P_h(h) + P_c(x), \tag{5}$$

since the steps may be divided between the hook and the coil in any fashion except for a single step hook.

If following the hooked coil the path enters an open hook, the probability of making an open hook of y steps is (as for the closed hook)

$$P_y(y) = (1/4)^y. \tag{6}$$

Since the hooked coil and the open hook may divide the total number of n between them in any fashion except for a single-step open hook, the probability of making a hook-coil-open-hook, which by our theorem is indeed the most general chain, is

$$P_{hcy}(n) = P_{hc}(n) + \sum_{x=2}^{n} P_{hc}(n-x)P_y(x). \tag{7}$$

Allowing now for the choice of initial direction, we have the probability of a chain of at least n steps

$$P_{ch}(n) = 2P_{hcy}(n), \tag{8}$$

and the probability of making a path of exactly n steps, *i.e.*, a path terminating on the nth step, is

$$P_p(n) = P_{ch}(n-1) - P_{ch}(n). \tag{9}$$

The final result can be stated as an iterated function. If we define

$$F_0(n) = \sum_{i=0}^{[n/3]} (1/2)^{n+3i} \frac{(n-2i)!}{(n-3i)!i!}, \tag{10}$$

and, by iteration,

$$F_{k+1}(n) = F_k(n) + \sum_{x=2}^{n} F_k(n-x)(1/4)^x, \tag{11}$$

then the probability that a path does not terminate by the nth step is $2F_3(n)$ and the probability that a path terminates in exactly n steps is

$$P_p(n) = 2[F_3(n-1) - F_3(n)]. \tag{12}$$

Table 1 gives the first few values of $P_{ch}(n)$ and $P_p(n)$.

TABLE 1

n	1	2	3	4	5	6	7
$P_{ch}(n)$	1	12/16	30/64	70/256	160/1024	360/4096	802/16384
$P_p(n)$	0	4/16	18/64	50/256	120/1024	280/4096	638/16384

FIGURES INSCRIBED IN CONVEX SETS

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We shall say that a regular polygon or polyhedron is inscribed in a convex set* X if every vertex of the polygon or polyhedron is a point of the frontier of X. It is known that it is possible to inscribe a square in any plane convex set (see [1], [3], [5], [7]) and that if a point p is given interior to a three-dimensional convex set X then it is possible to inscribe a square in X with p as its centre ([2]).

On the other hand it is known that about any convex set in n dimensions it is possible to circumscribe an n-dimensional cube, *i.e.* there are support hyperplanes of the convex set which form the faces of a cube ([4], [6]). The dual of this result in two dimensions or three dimensions asserts that it is possible to inscribe a square or a regular octahedron (as the case may be) in a central convex set.

The object of this note is to give examples which show that these results are, from certain points of view, the most that can be asserted. Examples are given with the following properties.

(A) A plane set of constant width† in which it is impossible to inscribe a regular n-gon with $n>4$.

(B) A three-dimensional convex set such that for every $n>4$ there is a point P interior to the set such that it is not possible to inscribe a regular n-gon in the set with P as circumcentre.

(C) A central three-dimensional convex set in which it is impossible to inscribe a cube.

* By "convex set" is always meant "bounded closed convex set." The containing space is real Euclidean space.

† A convex set is of constant width if the distance apart of two parallel support lines is the same whatever the direction of the lines.

Example (A). Let Γ be a Reuleaux triangle§ with vertices A, B, C and of width λ. Any n-gon with $n \geqq 5$ inscribed in Γ must have two vertices on the same arc AB, BC, or CA. Suppose that the n-gon $P_1P_2 \cdots P_n$ is such that P_1 and P_2 lie on BC.

Since BC is an arc of a circle whose centre is A the perpendicular bisector of the segment P_1P_2 passes through A. If n is odd it also passes through the vertex $P_{(n+3)/2}$ which, since it lies on the frontier of Γ lies either at A or on the subarc P_1P_2 of the arc BC. In the second case the polygon is not regular since the segment P_1P_2 would be of greater length than P_2P_3. In the first case, if the polygon is regular there are two vertices of it on each arc AB and AC of Γ. For either $P_1=B$ and $P_2=C$ or the length of segment P_1P_2 is less than λ. If the second alternative holds, the length of segments $P_{(n+3)/2}P_{(n+5)/2}$ and $P_{(n+1)/2}P_{(n+3)/2}$ are both less than λ; thus $P_{(n+5)/2}$ lies on arc AB and $P_{(n+1)/2}$ lies on arc AC. But then by the same argument as above the points B and C are vertices of the polygon. Now every vertex of a regular polygon with an odd number of vertices lies on the perpendicular bisector of a side of the polygon. But every side of our polygon is a chord of one of the arcs AB, BC, CA of Γ and this implies that the only possible positions for the vertices of the polygon are A, B, C. This is a contradiction with the fact that the n-gon has more than four vertices. Thus no regular n-gon with n odd, $n \geqq 5$ can be inscribed in Γ.

Next consider the case when n is even, $n \geqq 8$. Again a regular n-gon $P_1 \cdots P_n$ with three vertices on one of the arcs AB, BC, CA say P_1, P_2, P_3 on the arc BC is such that the centre of the n-gon (which lies on the perpendicular bisector of P_1P_2 and on the perpendicular bisector of P_2P_3) must lie at A and cannot therefore be contained in the Reuleaux triangle. Thus no regular n-gon with n even and $n \geqq 8$ can be inscribed in Γ.

Finally consider the case $n=6$. As above $P_1 \cdots P_6$ has at most two vertices on each arc AB, BC, CA and thus it must have exactly two vertices on each of these arcs, say P_1P_2 on BC, P_3P_4 on CA and P_5P_6 on AB. Let l denote the perpendicular bisector of P_1P_2. Then if P_1P_2 lie on BC, l passes through A. Further $P_1 \cdots P_6$ is symmetrical about l. If we reflect our figure in l, P_3 becomes P_6, P_4 becomes P_5. Hence the arc AB is the reflection of the arc AC in l, and l must bisect the arc BC. Similarly the perpendicular bisector of P_3P_4 passes through B. Thus the centre of the polygon coincides with the circumcentre of Γ and the regular hexagon that can be inscribed in Γ is unique.

We construct a set $\Gamma(x)$ by modifying Γ as follows. On the arc AC of Γ take a point A_1, near to A, at a distance of x from A and let A_2 denote the midpoint of the arc AB. Let D be a point of intersection of the two circles which have radius λ and centres A_1, A_2 and suppose that D is such that the circular arcs of radius λ, A_1A_2, A_2B, BD, DC, CA_1 bound a pentagon of constant width λ. (The circular arc A_1A_2 is part of the circle centre D and radius λ.) We denote this pentagon by $\Gamma(x)$.

§ A Reuleaux triangle is the common part of three closed circular discs of radius r, the three centres of which form an equilateral triangle of side r.

As x decreases to zero, $A_1 \to A$, $D \to C$ and the pentagon A_1A_2BDC tends to Γ. It follows that for any n, $n \geqq 7$ or $n = 5$ there exists a positive number δ_n such that for $0 \leqq x \leqq \delta_n$ it is impossible to inscribe a regular n-gon in $\Gamma(x)$. Further the only regular hexagons that can be inscribed in $\Gamma(x)$ must tend to the unique regular hexagon that can be inscribed in Γ, as $x \to 0$. Thus such a hexagon, if it exists, will have two vertices on arc A_1C, one each on arcs BA_2 and A_2A_1 and two on arc BD.

We show firstly that if x is small $\Gamma(x)$ contains no regular inscribed hexagon. Suppose that such a hexagon exists with vertices $P_1 \cdots P_6$ where P_1, P_2 lie on arc BD, P_3P_4 lie on arc CA_1, P_5 lies on arc A_1A_2 and P_6 lies on arc A_2B. The perpendicular bisector of segment P_3P_4 passes through B, and the hexagon is symmetrical about it. Thus if we reflect our figure in this line, P_6 reflects onto P_1 and thus arc BA is the reflection of arc BD. Hence arc A_2A_1 reflects into an arc which is entirely interior to $\Gamma(x)$ except for its end points. But arc A_2A_1 contains P_5 which is not one of its end points and which reflects onto P_2, a frontier point of $\Gamma(x)$. This contradiction shows that $\Gamma(x)$ contains no regular inscribed hexagon.

Take an integer N so large that a regular n-gon with $n \geqq N$ which is inscribed in a set of constant width has sides of length at most $\lambda/8$. In defining $\Gamma(x)$ choose x so small that the distance of A_1 from C is greater than $3\lambda/4$ and such that no regular n-gons can be inscribed in $\Gamma(x)$, where $n = 5, \cdots, N$. If a regular n-gon with $n > N$ could be inscribed in $\Gamma(x)$ there would be at least three vertices on arc A_1C (for otherwise either all vertices would lie on arcs BA_2 and BD or there would be two consecutive vertices at a distance apart greater than $\lambda/8$; neither of these alternatives is possible). But then if n is even, B is the centre of the polygon which is therefore not contained in $\Gamma(x)$, and if n is odd B is a vertex of the polygon which is equidistant from the three vertices on A_1C. This is impossible.

Thus finally $\Gamma(x)$ is a set of constant width in which it is impossible to inscribe any regular polygon with $n \geqq 5$.

Example (B). Let Y be a plane convex set satisfying the conditions of example (*A*) and let its circumcentre be k and let X be a right cylinder on Y as base. Let P_n be a sequence of points converging to k and contained in the interior of Y. For every fixed $n \geqq 5$ there is an integer m such that there is no regular n-gon inscribable in X with P_m as circumcentre.

Suppose that this is not the case. Then by the Blaschke selection theorem we can select a subsequence P_{n_i} such that each P_{n_i} is the circumcentre of a regular n-gon, say T_{n_i}, inscribed in X and convergent to a set T. T is either a frontier point of X or a regular n-gon whose circumcentre is k. But in the second case since $T \subset X$ we must have T contained in Y. Thus there would be a regular n-gon inscribed in Y. We know that this is impossible. In the first case the dimensions of T_n tend to zero as i tends to infinity. But each T_m must contain a vertex not on Y and thus at a fixed positive distance say τ from k. Hence as

$P_{n_i} \to k$ the linear dimension of T_{n_i} is bounded below and the first alternative cannot occur either.

Thus for every $n>4$ there is a point P interior to X such that no n-gon can be inscribed in X whose circumcentre coincides with P.

Example (C). Of the eight vertices of a cube the set of six points obtained by omitting two diametrically opposite vertices is called a V set. We need the following lemma.

LEMMA. *If the points of a V set belong to the frontier of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{1}$$

where no two of the positive real numbers a, b, c are equal, then they are of the form $(\pm d, \pm d, \pm d)$ where any choice of sign is possible and where d satisfies

$$d^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=1.$$

Let the center of the V set be (f, g, h); then the points of this set are of the form $(f+k_i, g+l_i, h+m_i)$ and $(f-k_i, g-l_i, h-m_i)$, $i=1, 2, 3$. Further the points of V do not lie in a plane through (f, g, h). Thus

$$\begin{vmatrix} k_1 & l_1 & m_1 \\ k_2 & l_2 & m_2 \\ k_3 & l_3 & m_3 \end{vmatrix} \neq 0.$$

But since the points of the V set lie on the frontier of the ellipsoid, simple algebraic manipulations lead to

$$f\frac{k_1}{a^2}+g\frac{l_1}{b^2}+h\frac{m_1}{c^2}=0,$$

$$f\frac{k_2}{a^2}+g\frac{l_2}{b^2}+h\frac{m_2}{c^2}=0,$$

$$f\frac{k_3}{a^2}+g\frac{l_3}{b^2}+h\frac{m_3}{c^2}=0.$$

Hence $f=g=h=0$.

Next suppose for definiteness that $a>b>c>0$ and let V_1 be the V set formed from the six points $(d, d, -d)$, $(-d, -d, d)$, $(-d, d, -d)$, $(d, -d, d)$, $(-d, d, d)$, $(d, -d, -d)$. Then any ellipsoid which has the origin as centre and in which V_1 is inscribed is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\lambda(yz+xz+xy)=1-\lambda d^2, \tag{2}$$

where $\lambda d^2 < 1$. The points $(a(1-\lambda d^2)^{1/2}, 0, 0)$, $(0, 0, c(1-\lambda d^2)^{1/2})$ satisfy (2). Thus if the lengths of the semimajor and semiminor axes of (2) are a' and c', respectively, then $a' \geq a(1-\lambda d^2)^{1/2}$, $c' \leq c(1-\lambda d^2)^{1/2}$. Moreover, equality holds only if the tangent plane to (2) at $(a(1-\lambda d^2)^{1/2}, 0, 0)$ is parallel to the plane $x=0$ and this is so if and only if $\lambda = 0$. Thus if $\lambda \neq 0$ then $a'/c' > a/c$. Hence no ellipsoid of the form (2) is similar to the ellipsoid (1) unless it is identical with it. Since any two V sets are similar, it follows that no V set can be inscribed in (1) except the V set V_1 or a V set obtained from V_1 by a similarity transformation which transforms (1) into itself. Such a transformation transforms the set of points $(\pm d, \pm d, \pm d)$ into itself. This proves the lemma.

We now denote the points on or inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

by E and obtain the required set by a modification of E. Let $\pi(x)$ be the plane parallel to the tangent plane to E at (d, d, d) which passes through (x, x, x). When $x \neq 0$ denote the closed half space bounded by $\pi(x)$ and containing the origin by $L(x)$. Write $E(x) = E \cap L(x) \cap L(-x)$.

Let x approach d by values less than d, say in the sequence $\{x_i\}$. If there is a cube C_i inscribed in $E(x_i)$ then as $i \to \infty$, C_i must approach the cube with vertices $(\pm d, \pm d, \pm d)$ since this is the only cube inscribable in E. Thus for i sufficiently large C_i has one vertex each on $\pi(x)$ and $\pi(-x)$ and the remaining six on E, *i.e.* there is a V set of six vertices of C_i on E. By the lemma this implies that C_i is the cube with vertices $(\pm d, \pm d, \pm d)$. But this is impossible since the set $E(x_i)$ does not contain the point (d, d, d) or the point $(-d, -d, -d)$.

Thus finally if i is large, no cube can be inscribed in $E(x_i)$. Since $E(x_i)$ is central we have established the third example.

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BRINGING CALCULUS UP-TO-DATE

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1. Introduction. There are many conversations, committee meetings, *etc.*, today about the modernization of the undergraduate calculus course; but all too often the attack on the problem falls short of being comprehensive. Calculus has been in cold storage for over fifty years now. These have been highly productive years in mathematics, and the result is that more changes are in order than most people would like to admit.

The relevant branches of modern mathematics would seem to be real function theory and differential geometry, and a survey of these fields suggests at least the five points listed below. It is significant that point (i) can be gleaned from a study of real function theory and is already coming into vogue in calculus, while points (ii)–(v) come largely from differential geometry and are too generally overlooked. Calculus is too often erroneously classified as a part of real function theory only.

(i) It is essential to distinguish between a function f and its values $f(a)$.

(ii) The coordinate variables x and y of analytic geometry are themselves mappings; x maps points into their abscissas; y maps points into their ordinates.

(iii) There is an important difference in point of view toward coordinate variables in analytic and differential geometry. In analytic geometry x and y are defined over the entire plane, and $y=f(x)$ is a conditional equation describing a locus. In differential geometry the locus is preassigned; x and y are restricted to this locus, and $y=f(x)$ is an identity. A calculus problem starts as analytic geometry and finishes as differential geometry. Thus, somewhere in the middle the symbols change meaning. (More details on this point in Section 2.)

(iv) Modern differential geometry has finally produced a definition of the differential that is quite satisfactory for purposes of integration and differentiation theory. In this definition the concept of differential has been divorced from that of approximate increment. (Details in Section 3.)

(v) In the modern, advanced study of integration of differential forms the theory of the integral is lifted from real function theory, and the algebra of the differential forms is shown to fit into this framework. When this abstract theory is specialized to the simplified, concrete cases discussed in calculus, modern differential theory becomes an effective (and simpler) tool for certain developments in the theory of the integral. (See Sections 5 and 6.)

2. Functions and variables. To begin with, define the word *function* to mean a mapping of numbers into numbers. A later generalization will include

mappings of n-tuples of numbers into numbers. This is the traditional use of the word function in calculus.

One also encounters spaces whose elements are called points. These take the form of lines, planes, curves, surfaces, 3-space, space-time, *etc.* An essential part of the machinery is the mapping of these points into numbers. Let these mappings be called *variables*. Linguistically incongruous as it may seem, this is traditionally the most common use of the word variable in calculus. This definition follows the principle that language is established by usage, not by logic. It retains the word and gives a more enlightened description of what it has been used to mean. It should be noted that probability theory has already established this usage with the phrase *random variable*.

The coordinate variables x and y are variables in the sense just defined. In plane analytic geometry each of these symbols stands for a mapping of the entire plane onto the real number system. If x has this meaning and f is a function, then $f(x)$ is used to denote a composite mapping:

$$\text{point} \xrightarrow{x} \text{number} \xrightarrow{f} \text{another number.}$$

The form

$$y = f(x) \tag{1}$$

is now a *conditional equation*. Here there are two symbols, y and $f(x)$, each standing for a mapping; but in analytic geometry (1) is not regarded as an assertion that these are two names for the same mapping. Rather, (1) appears (or should appear) only as a noun clause in the phrase, "the locus of $y=f(x)$." This locus is, of course, the set of all points p in the common domain of x and y for which $y(p)=f[x(p)]$. Note that this last "$=$" means what it should.

Suppose, now, that (1) has been given in proper context and its locus has been found and named C. The next step in the logical analysis of a calculus problem is to restrict the mappings x and y to C. Note that this could not be done before because the superstructure described above was used to define C. It would help if the restricted mappings were given new names; say, start with X and Y and boil them down to x and y. However, this is probably asking for too much of a change in set habits. In any case, with x and y restricted to C, (1) becomes an *identity*—an assertion that two mappings are the same. At this stage one has a special case of the type structure studied in modern differential geometry and so can turn to the literature of that discipline for further enlightenment. What follows is an informal summary of the specialization of this work to calculus. For a more complete discussion of the general theory see, for example, Chevalley, *Theory of Lie Groups*, Princeton, 1946.

3. Differentials. A *one-dimensional manifold* is a structure consisting of a point set C and a set of *coordinate variables* defined thereon. Precise postulates can be given, but a very loose description will suffice for the present discussion. The set C is a smooth curve, and each of the coordinate variables is continuous

and locally one-to-one. Since each is locally one-to-one, each pair u, v is related at least locally by an identity of the form $v=f(u)$, and it is convenient to assume that each such connecting function f has three derivatives. Note that a locus in the plane with the rectangular coordinate variables restricted to it will generally form such a structure.

Consider three classes of variables on C.

X: all variables on C,
Y: those related to a coordinate by a differentiable function,
Z: those related to a coordinate by a thrice-differentiable function.

Let D be a mapping of Y into X, and let Du denote the map of u by D. If p is a point on the curve, denote the value of Du at p by $(Du)_p$. Such a mapping D is called a *derivative operator* at p provided

$$\text{(a)}\quad [D(au+bv)]_p = a(Du)_p + b(Dv)_p,$$
$$\text{(b)}\quad [D(uv)]_p = u(p)(Dv)_p + v(p)(Du)_p.$$

Here u and v are variables and a and b real numbers.

Regarding a constant a as a variable, observe that

$$(Da)_p = 0, \tag{2}$$

because by (a), $[D(au)]_p=a(Du)_p$, and by (b), $[D(au)]_p=a(Du)_p+u(p)(Da)_p$; and if u is chosen so that $u(p)\neq 0$, then (2) follows. Observe also that setting $u=v$ in (b) yields

$$[D(u^2)]_p = 2u(p)(Du)_p. \tag{3}$$

Now, let u be a coordinate, and let $v=f(u)$. Assume f has three derivatives; then there is a differentiable function g such that

$$v = v(p) + f'[u(p)][u-u(p)] + g(u)[u-u(p)]^2. \tag{4}$$

Recalling (2) and (3), operate on (4) with D at p to obtain

$$(Dv)_p = f'[u(p)](Du)_p. \tag{5}$$

Each of the other terms on the right vanishes either because of (2) or because it contains the factor $u(p)-u(p)$. Note that in this step D must be defined on Y because it must operate on $g(u)$. However, (5) is proved only for $v\in Z$ because this is required in order to set up (4).

Let D_u be the operator such that $D_u u=1$ at each point on the curve. By (5) this determines D_u on v where $v=f(u)$, and indeed

$$D_u v = f'(u). \tag{6}$$

The variable $D_u v$ is called the *derivative of v with respect to u*. The notation f' denotes a purely function theoretic concept; f' is defined from f in terms of limits of difference quotients. On the other hand, D_u is an operator defined by purely

algebraic conditions. Thus, (6) is not a definition; it is a theorem relating these two basically different notions.

Consider now D_1 and D_2, two derivative operators at p. If for some variable u, $(D_1u)_p = a(D_2u)_p$, then for $v = f(u)$,

$$(D_1v)_p = f'[u(p)](D_1u)_p = af'[u(p)](D_2u)_p = a(D_2v)_p.$$

That is, at p any two derivative operators are constant multiples one of the other. So, the set of derivative operators at p has the algebraic structure of a straight line. Identify it with T_p, the tangent line to the curve at p, and define the *tangent bundle* for the given curve as the set of all ordered pairs (p, D) where p is a point of the curve and D is a derivative operator at p.

Finally, the *differential* is defined as follows. For u a variable on the curve, du is a variable on the tangent bundle. Its value at (p, D) is denoted by $du_p(D)$, and du is defined by

$$du_p(D) = (Du)_p. \tag{7}$$

Substitute (6) into (5) to obtain

$$(Dv)_p = (D_uv)_p(Du)_p. \tag{8}$$

By (7), $(Dv)_p$ and $(Du)_p$ are values of appropriate differentials; so (8) may be written

$$dv_p(D) = (D_uv)_p du_p(D).$$

This is for an arbitrary p on the curve and D on T_p; so

$$dv = D_uv\,du \tag{9}$$

over the tangent bundle. An appropriate name for (9) is "Fundamental Theorem on Differentials." At any rate, in the modern theory it is a theorem, not a revolving definition that changes meaning with every change of "independent" variable.

Multidimensional cases follow the same pattern. Let C be an n-dimensional manifold—intuitively, a uniformly n-dimensional, smooth set with variables attached. The basic form relating variables will be $v = f(u_1, \cdots, u_n)$. Derivative operators are defined by the same postulates as before, and by the n-dimensional generalization of (4) it is shown that they form a family of n-dimensional vector spaces. A derivative operator is determined if specified on n variables, and the operators $\partial/\partial u_i$ defined by

$$\frac{\partial u_j}{\partial u_i} = \begin{cases} 1 & \text{for } j = i, \\ 0 & \text{for } j \neq i \end{cases}$$

form a basis in the tangent bundle. Differentials are defined again by (7), and the fundamental theorem reads

$$dv = \sum_{i=1}^{n} \frac{\partial v}{\partial u_i}\, du_i. \tag{10}$$

4. Some immediate advantages. The ultimate advantage of saying things correctly rather than incorrectly need not be discussed. Let it be understood, then, that the following list—which is quite probably nonrepresentative—merely points out a few of the common difficulties that can be cleared up by the modern approach.

(a) *Differentials identified with variables.* A variable v on C generates a differential dv on the tangent bundle, and this differential retains its identity whether $v=f(u)$ or $v=g(w)$. Note that df and dg are not defined. Understanding of this will help to clear up many notation obscurities.

(b) *Second differentials.* No one seems to know exactly what these ought to be. Function theorists sometimes want $d^2v=D_u^2v du^2$; at other times (when changes of variable are in order) they want $d^2v=D_u^2v du^2+D_uv d^2u$. In working, say, with parametric equations the student is apt to use the first of these forms as though it had the invariance properties of the second. It seems advisable to leave these out of calculus completely, and the development outlined above does this very nicely. There the differential of dv is not defined because the domain of dv is the tangent bundle while the differential is defined only for a variable whose domain is C.

(c) *Chain rules.* Given $f(x, y, u, v)=g(x, y, u, v)=0$, it is common practice to find, say, $\partial u/\partial x$ in terms of other partial derivatives by writing out various expressions modeled after (10), making substitutions and equating coefficients of an appropriate differential. This is an effective technique for deriving chain rules, and in the modern theory it is quite acceptable. However, if (10) is a definition rather than a theorem, it must be proved invariant under coordinate changes. One must often know that the differential form is invariant under the very chain rule he is seeking to find!

(d) *Increments and approximations.* The undergraduate is afraid to use the differential—in such manipulations as (c), for example—because of the approximation bugaboo introduced with the now outmoded definition. In the modern theory this fear is never introduced, and the approximate increment problem is properly classified—under Taylor's theorem.

5. Line and surface integrals. Let u and v be variables on a curve C. Partition C by points $p_0, p_1, \cdots, p_n$. For each i, let q_i be the point on the tangent line T_{p_i} whose distance from p_i is the length of the arc p_ip_{i+1} on C. Intuitively, "unroll" each increment of arc onto a tangent line. As noted in Section 3, the set of derivative operators at p_i forms a "line," to be identified with the tangent line T_{p_i}. To obtain the usual geometric interpretation, specify that if s is the arc length variable on C, then D_s is identified with the unit tangent vector. In any

case, each point q_i constructed above is identified with a derivative operator at p_i; so the notation $dv_{p_i}(q_i)$ has an obvious meaning. Now, form sums

$$\sum_{i=0}^{n-1} u(p_i)dv_{p_i}(q_i). \tag{11}$$

Take an appropriate limit of these, and call the result

$$\int_C u dv. \tag{12}$$

The important feature to this development of the line integral is that the points q_i on the tangent lines are determined by an intrinsic notion (arc length) on C and have no connection with the variables u and v. Thus, the substitution theorem, $\int_C udv = \int_C uD_w v dw$, is an immediate consequence of the fundamental theorem on differentials (9). The approximating sums (11) are invariant under the substitution; hence so is the integral. The familiar $\int_a^b f(x)dx$ is now a special case of (12)—in which C is an interval on the x-axis. In this development the differential is clearly the same thing in differential and integral calculus.

In a double integral—carefully to be distinguished from an iterated integral —*exterior multiplication* of differentials must be introduced. This is an operation denoted by $*$ and characterized by the rule

$$du * dv = -dv * du. \tag{13}$$

This stems from the fact that $du * dv$ is to be a signed measure on oriented plane regions. Setting $u = v$ in (13) yields the corollary $du * du = 0$. Using these rules of multiplication (plus a distributive law) and substituting from the two-dimensional case of (10), one has

$$\begin{aligned} du * dv &= \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) * \left(\frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy\right) \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx * dx + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\,dx * dy \\ &\quad + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\,dy * dx + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dy * dy \\ &= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) dx * dy \\ &= \frac{\partial(u, v)}{\partial(x, y)}\,dx * dy. \end{aligned} \tag{14}$$

In general, a pair (du_p, dv_p) forms a set of affine coordinates in the tangent plane T_p. If the pair is orthogonal (and some pairs always are), then multipliers

a and b can be chosen so that (adu_p, bdv_p) forms a set of rectangular coordinates in T_p. In general, a and b depend on p; for example, on the tangent planes to the unit sphere $(d\phi, \sin\phi\, d\theta)$ generates rectangular coordinates. In any case, if (adu_p, bdv_p) is a rectangular set on T_p, and if A is a region in T_p positively oriented with respect to this coordinate system, then define $ab(du * dv)_p(A)$ to be the area of A. Since the Jacobian of a rotation is unity, (14) shows that this is consistent for all such rectangular sets. Values of other exterior product operators are then determined by (14).

Now, the substitution rule for multiple integrals consists merely of making the substitution (14) behind the integral sign. A definition of the surface integral patterned after that given above for the line integral makes this remarkably easy to justify.

Take a smooth surface S and partition it in an arbitrary fashion—not necessarily following coordinate lines—into subsets S_i. At some point p_i in S_i take a tangent plane. Here a minor complication appears; one cannot "unroll" the S_i as he did the arcs and preserve measure. However, define surface area—there are many tricks for accomplishing this—and construct on each tangent plane a set having the same area as S_i. Call this plane set A_i. The shape of A_i is immaterial because (see definition above) $(du * dv)_{p_i}(A_i)$ depends only on the area and orientation of A_i, not on its shape. Now, form sums

$$\sum_{i=1}^{n} w(p_i)(du * dv)_{p_i}(A_i). \tag{15}$$

Take the usual limit, and call it

$$\iint_S w du * dv.$$

Since the A_i depend in no way on u and v, the sums (15) are invariant under the substitution (14); and there emerges the familiar result

$$\iint_S w du * dv = \iint_S w \frac{\partial(u, v)}{\partial(x, y)}\, dx * dy.$$

6. Theorems of Green, Gauss, and Stokes. This view of the multiple integral brings an elegant unification to the theory relating an integral over a manifold to one over the boundary. There is a master formula:

$$\int \cdots \int_B u dv_1 * \cdots * dv_k = \iint \cdots \int_M du * dv_1 * \cdots * dv_k \tag{16}$$

where M is an oriented $(k+1)$-dimensional manifold and B is its boundary. This is proved in the usual way by reducing the integral on the right to an iterated integral and performing one integration.

Now, if A is a region in the plane and C its boundary, then (16) yields

$$\int_C (u\,dx + v\,dy) = \iint_A (du * dx + dv * dy).$$

However,

$$du * dx = \left[\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right] * dx = -\frac{\partial u}{\partial y}\,dx * dy,$$

$$dv * dy = \left[\frac{dv}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy\right] * dy = \frac{\partial v}{\partial x}\,dx * dy;$$

so the result is Green's theorem. A similar maneuver with the form

$$\int_C (u\,dx + v\,dy + w\,dz)$$

where C is an appropriate space curve yields the classical theorem of Stokes. The divergence theorem is also a special case of (16) as is easily shown by direct computation. Minus signs do not appear in the divergence theorem because in all this work the permutations of differentials are cyclic permutations, and a cyclic permutation of three things is an even permutation.

When presented in this light, all these theorems appear as generalizations of the fundamental theorem of calculus. For, this latter may be written $u(q) - u(p) = \int_p^q du$, and this is a specialization of (16) in which the difference on the left is regarded as an integral over the zero-dimensional boundary consisting of two points. Subtraction results because the boundary is considered oriented.

7. Notation. Roughly, the structure studied in calculus seems to consist of mappings (variables) "up" from point sets into the real number system together with other mappings (functions) "across the top." It is well to preserve this stratification in the notation, and the usual use of f and g for functions and the last letters of the alphabet for variables is all to the good. However, there are other distinctions that common notational practice fails to make.

The derivative of a function is another function whose values are appropriate limits of difference quotients. Note that a function is not differentiated "with respect to" anything in particular. Thus, an appropriate notation for the derivative of f is f', and it would be well if the prime symbol were reserved for functions only.

Derivatives of variables are generated by the derivative operators introduced above, and the basic form is $D_x y$. The differential form dy/dx is not merely an alternative to this. It means $dy \div dx$, but it follows at once from the fundamental theorem on differentials that $dy/dx = D_x y$.

Hybrid notation such as y' and df/dx abounds in the literature, but it is not really well-defined. One step in the modernization of calculus is to break some well-formed (too well-formed) habits in this respect. Frequently, the confusion stems from writing $y = f(x)$ and then confusing the symbols y and f. As for habits,

however, it is sad to relate that the author of a recent textbook made essentially the comment just made here about confusing y and f and then proceeded to do exactly that three pages later.

Similar distinctions need to be made in the multidimensional cases. If f maps n-tuples of numbers into numbers, its arguments are distinguished by position only. For example, $f(a, b) = a - b$ and $f(b, a) = b - a$ define the same function f. Thus, f_i is appropriate notation for the partial derivative of f with respect to its ith argument. As noted above, $\partial/\partial x$ is a derivative operator in manifold theory and should be reserved for application to variables.

8. Existence theorems. Little has been said so far about the role of real function theory in a program of modernizing calculus. In a sense this is a question quite independent of the ones raised so far here, but certainly the following question is pertinent to the present discussion. The algebraic theory of derivative operators and differentials outlined above yields formulas and techniques in a very elegant fashion, but it certainly is not self contained. Is any additional real function theory required as a background to tighten up the logic in this development?

In integral calculus one must develop the notion of integral of a function alongside that of integral of a variable (outlined above). The two are fairly easily related, and the usual existence proof appears in the function theory.

In differential calculus the existence questions are answered *a priori* in the postulates for a manifold. That is, differentiability conditions are imposed by fiat on the connecting functions in such a way that the algebraically defined derivative operators do represent differentiations in the function theoretic sense. However, to keep from operating in a vacuum, one must show that certain things are manifolds.

For purposes of calculus one gets an adequate supply of manifolds by taking the rectangular coordinates in $(n+k)$-space and considering the locus of n equations in them. The implicit function theorem (a standard item in advanced calculus) gives conditions under which k of these variables become local coordinates on the locus, and the Brouwer theorem on invariance of domain (now appearing in the better advanced calculus texts) guarantees that the locus has the proper dimension.* To verify the hypotheses of these theorems, one must know the differentiability properties of the elementary functions. These may be established in the usual way through theorems on differentiability of sums, products, composites, *etc.*

It is probably fair to say, then, that the introduction of the modern theory of the differential changes very little the number or the nature of the real function theory existence proofs required for a logically sound development of calculus. This, of course, is subject to the stipulation that calculus is restricted to

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* Added in proof: A recent note by Yamabe, this MONTHLY, vol. 64, 1957, p. 725, gives precisely the result needed.

the well-behaved cases. A general attack on the topological problems of manifold theory is much more difficult.

9. Two facets to the problem. Most programs for the improvement of calculus amount to the injection into the course of more and better real function theory. It is the purpose of the present paper not to discourage this, but to suggest that a more imperative project is the injection of better—if not more—differential geometry. This latter is more imperative because the garden variety calculus contains more downright errors in differential geometry than in real function theory. On the other hand, it is more difficult for at least two reasons. First, to bring the differential geometry up to date will change the appearance of a calculus text; it will modify definitions, terminology, notation, procedures. Second, while most calculus teachers are well grounded in real function theory, for many of them graduate study came before some of the important developments in differential geometry.

Despite these difficulties an effort should be made. In failing to bring calculus up to date we are transmitting to the next generation not the information available to our contemporaries, but that available to our grandfathers.

ON THE COEFFICIENTS OF RECIPROCAL POWER SERIES

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1. Introduction. Suppose that the two sequences of numbers $\{u_n\}$ and $\{f_n\}$ satisfy the relations

$$u_n = \sum_{i=0}^{n-1} u_i f_{n-i} + \delta_{0n}, \qquad n = 0, 1, \cdots, \tag{1}$$

where

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

Upon taking the generating functions of the sequences,

$$U(z) = \sum_{n=0}^{\infty} u_n z^n \quad \text{and} \quad F(z) = \sum_{n=1}^{\infty} f_n z^n,$$

relation (1) may be re-expressed as

$$U(z) = U(z)F(z) + 1, \tag{2}$$

where the power series and their multiplication are to be interpreted in a purely formal way. If $U(z)$ and $F(z)$ satisfy (2), then we shall say that $F(z)$ is the re-

ciprocal series determined by $U(z)$ (despite that in the usual sense $U(z)$ and $1-F(z)$ are reciprocals).

It is clear from (1) that either of the sequences $\{u_n\}$ or $\{f_n\}$ determines the other uniquely. It is also easy to see that if every $f_n \geqq 0$, then also each $u_n \geqq 0$ (but not conversely). Theodor Kaluza in [3] has studied the problem of deciding which sequences $\{u_n\}$ determine sequences $\{f_n\}$ such that each $f_n \geqq 0$, and it is also to this problem that the present paper is devoted. In Section 2 a necessary and sufficient condition for $f_n \geqq 0$ (all n) is derived. Although it is cumbersome, this condition has several interesting consequences, and these are discussed in Section 3.

Equation (1) has an important probability interpretation. If E is a "recurrent event" in the sense described in [1] (Ch. 12), then the quantities u_n, equal to the probability that E occurs at time n, and f_n, equal to the probability that E occurs for the first time at time n, satisfy (1). This situation supplies a motivation for most of the results to follow, but will not be made use of explicitly. In the probability interpretation it is of course necessary that $\sum f_n \leqq 1$, while in this paper the f_n need not be bounded.

Before proceeding to the basic theorem, examples may be mentioned with $f_n \geqq 0$, where $U(z)$ is known explicitly and is other than a rational function. One such is

$$U(z) = (1 - az^2)^{-1/2}, \qquad F(z) = 1 - (1 - az^2)^{1/2}, \qquad a > 0.$$

These functions occur in a gambling problem ([1], p. 246). Another example* is

$$U(z) = (\tan z)/z, \qquad F(z) = 1 - z \cot z = -\sum_{k=1}^{\infty} (-4)^k B_{2k} z^{2k}/(2k)!$$

where B_{2k} are Bernouilli numbers; still another is given in [3], Theorem 2.

2. A condition for $f_n \geqq 0$.

THEOREM 1. *The sequence $\{f_n\}$ obtained from (1) satisfies $f_n \geqq 0$ for all n if and only if there exists a matrix $P = [p_{ij}]$ with $p_{ij} \geqq 0$ and only a finite number of nonzero terms in each row, such that*

$$u_n = p_{11}^{(n)}, \quad \textit{where} \quad P^n = [p_{ij}^{(n)}]. \tag{3}$$

Proof. Let $P = [p_{ij}]$ be any matrix with a finite number of nonzero terms in each row. Then multiplication is well-defined even if the matrix is infinite, and

$$p_{11}^{(n)} = \sum_{i_1} \cdots \sum_{i_{n-1}} p_{1i_1} p_{i_1 i_2} \cdots p_{i_{n-1}1}. \tag{4}$$

Define

$$r_n = \sum_{i_1 \neq 1} \cdots \sum_{i_{n-1} \neq 1} p_{1i_1} p_{i_1 i_2} \cdots p_{i_{n-1}1}. \tag{5}$$

* This example was suggested by the referee.

Now let the terms making up the sum in (4) be partitioned into classes E_j, where $p_{1i_1}\cdots p_{i_{n-1}1}\in E_j$ if $i_1\neq 1,\cdots,i_{j-1}\neq 1$ and $i_j=1$, and $p_{1i_1}\cdots p_{i_{n-1}1}\in E_n$ if $i_1,\cdots,i_{n-1}\neq 1$. Then if $j=n$,

$$\sum_{E_n} p_{1i_1}\cdots p_{i_{n-1}1}=r_n$$

by definition, while if $j<n$,

$$\sum_{E_j} p_{1i_1}\cdots p_{i_{n-1}1} = \left\{\sum_{i_1\neq 1}\cdots\sum_{i_{j-1}\neq 1} p_{1i_1}\cdots p_{i_{j-1}1}\right\}\left\{\sum_{k_1}\cdots\sum_{k_{n-j-1}} p_{1k_1}\cdots p_{k_{n-j-1}1}\right\}=r_j p_{11}^{(n-j)}. \tag{6}$$

Summing (6) over j yields, for $n>0$,

$$p_{11}^{(n)}=\sum_{j=1}^{n}\sum_{Ej} p_{1i_1}\cdots p_{i_{n-1}1}=\sum_{j=1}^{n-1} r_j p_{11}^{(n-j)}+r_n,$$

or, letting $p_{11}^{(0)}=1$,

$$p_{11}^{(n)}=\sum_{j=0}^{n-1} r_{n-j}p_{11}^{(j)}+\delta_{0n}. \tag{7}$$

It follows from (7), since solutions to (1) are unique, that if $p_{11}^{(n)}=u_n$, then $r_n=f_n$. If, therefore, all $p_{ij}\geqq 0$, it is obvious from (5) that $f_n\geqq 0$.

To prove the necessity of the condition, it is only required to exhibit a suitable matrix P. Suppose that $\{u_n\}$ does determine $f_n\geqq 0$, and let

$$P=\begin{bmatrix} f_1 & 1 & 0 & 0 & 0\cdots \\ f_2 & 0 & 1 & 0 & 0\cdots \\ f_3 & 0 & 0 & 1 & 0\cdots \\ f_4 & 0 & 0 & 0 & 1\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot\cdots \end{bmatrix}.$$

It is easy to see from (5) that $r_n=f_n$; therefore from (7) it follows that $p_{11}^{(n)}=u_n$. This completes the proof.

3. Consequences. Although Theorems 2 and 3 are special cases of Theorem 4, they precede it for reasons of aesthetics.

THEOREM 2. *Suppose $U(z)$ and $V(z)$ are formal power series with $u_0=v_0=1$ such that their reciprocal power series have nonnegative coefficients. Then $U(z)*V(z)=\sum_{n=0}^{\infty}u_nv_nz^n$ has this property also.*

Proof. Let P and Q be matrices with nonnegative entries such that $u_n=p_{11}^{(n)}$ and $v_n=q_{11}^{(n)}$ (Theorem 1); let $R=P\otimes Q$ be the Kronecker product of P and Q (see for instance [2], pp. 176–179). Then since $R^n=P^n\otimes Q^n$,

$$r_{11}^{(n)} = p_{11}^{(n)} q_{11}^{(n)} = u_n v_n.$$

But R also has nonnegative entries and finitely many nonzero terms in each row. Another application of Theorem 1 therefore completes the proof.

THEOREM 3. *Let $U(z)$ and $V(z)$ be as in Theorem 2. Then $W(z) = \sum_{n=0}^{\infty} w_n z^n$, where $w_n = \sum_{i=0}^{n} \binom{n}{i} u_i v_{n-i}$ and $w_0 = 1$, also determines a reciprocal series with nonnegative coefficients.*

Proof. Choose P and Q as in the proof of Theorem 2. Let I_1 and I_2 be identity matrices with the same number (possibly infinite) of rows and columns as P and Q, respectively. Let $A = P \otimes I_2$, $B = I_1 \otimes Q$, and $C = A + B$. Then A and B commute, so that

$$C^n = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i} = \sum_{i=0}^{n} \binom{n}{i} (P^i \otimes I_2)(I_1 \otimes Q^{n-i}) = \sum_{i=0}^{n} \binom{n}{i} P^i \otimes Q^{n-i},$$

which implies $c_{11}^{(n)} = w_n$. Since C satisfies the conditions of Theorem 1, this proves Theorem 3.

A more general result, which in a sense gives a connection between the Hadamard product (*) of Theorem 2 and the "product" discussed in Theorem 3, is the following:

THEOREM 4. *Let $U(z)$ and $V(z)$ be as above and let $W(z) = \sum_{n=0}^{\infty} w_n z^n$, where $w_0 = 1$ and*

$$w_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} \alpha^i \beta^j \gamma^k u_{i+j} v_{i+k},$$

with α, β, and γ nonnegative real numbers. Then $W(z)$ determines a reciprocal series with nonnegative coefficients.

Proof. By means of the matrix

$$D = \alpha(P \otimes Q) + \beta(P \otimes I_2) + \gamma(I_1 \otimes Q),$$

the proof may be carried out similarly to that of Theorem 3. Note that 0^n is to be interpreted as δ_{0n}, so that putting $\alpha = 1$, $\beta = \gamma = 0$ gives Theorem 2, while $\alpha = 0$, $\beta = \gamma = 1$ yields Theorem 3.

Also readily obtainable is a necessary condition:

THEOREM 5. *If $U(z)$ has a reciprocal series $F(z)$ with nonnegative coefficients, then $u_{n+m} \geqq u_n u_m$ for all n and m.*

Proof. Let the matrix P satisfy $p_{11}^{(n)} = u_n$ as guaranteed by Theorem 1. Since $p_{11}^{(n)} \geqq 0$,

$$u_{n+m} = p_{11}^{(n+m)} = \sum_i p_{1i}^{(n)} p_{i1}^{(m)} \geqq p_{11}^{(n)} p_{11}^{(m)} = u_n u_m.$$

The class of formal power series with reciprocals having nonnegative coefficients is a semigroup under Hadamard multiplication (*) by Theorem 2. From Theorem 5 it is easy to describe the units of this semigroup:

COROLLARY. *$U(z)$ and $U^*(z)=\sum_{n=0}^{\infty}(1/u_n)z^n$ both have reciprocal series with nonnegative coefficients if and only if $U(z)=\sum_{n=0}^{\infty}(az)^n$ for some $a>0$.*

Proof. The "if" part is obvious. To show necessity, first note that $U^*(z)$ is defined if $u_n\neq 0$ for all n, which by Theorem 5 occurs when $u_1\neq 0$. Also by Theorem 5 the coefficients of U and U^* must satisfy respectively

$$u_n \geqq u_1^n \quad \text{and} \quad \frac{1}{u_n} \geqq \left(\frac{1}{u_1}\right)^n,$$

and so $u_n=u_1^n$ with $u_1=a>0$.

To conclude, we shall state a theorem which is proved in [3], and use it to obtain a result on nonformal series.

THEOREM (Kaluza). *Suppose that $u_0=1$, $u_1>0$ and that*

$$\begin{vmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{vmatrix} \geqq 0, \qquad n \geqq 1.$$

Then if $\{f_n\}$ and $\{u_n\}$ are related by (1), (i.e., are the coefficients of reciprocal power series), $f_n\geqq 0$ for all n.

COROLLARY. *If $U(z)=\sum_{n=0}^{\infty}u_nz^n$ has a zero inside its (assumed nonnull) convergence circle and u_n are all real with $u_0\neq 0$, then for some $k\geqq 1$, $u_k^2>u_{k-1}u_{k+1}$.*

The idea of the proof of the corollary is that if the conclusion failed, the reciprocal series $F(z)$ to $(1/u_0)U(z)$ would have $f_n\geqq 0$. In this case, by (1), $f_n \leqq u_n/u_0$ so that the radius of convergence of F is not less than that of U. Then $(1/u_0)U(z)=1/(1-F(z))$ cannot vanish inside its convergence circle. This same idea, together with our Theorem 2, yields

THEOREM 6. *Let $W(z)=U(z)*V(z)$, where u_n and v_n are all real and $u_0\neq 0\neq v_0$. Then if $W(z)$ has a zero inside its convergence circle, for some $n\geqq 1$ either*

$$u_n^2>u_{n-1}u_{n+1} \text{ or } v_n^2>v_{n-1}v_{n+1}.$$

This theorem is also true if the Hadamard product $U*V$ is replaced by the more general "product" of Theorem 4.

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INVERSES OF VANDERMONDE MATRICES

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1. Introduction. In a recent paper [1], one finds explicit formulas for the derivatives of a polynomial $y=f(x)$ of degree n in terms of its values $y_i=f(x_i)$ at $n+1$ points defined by $x_i=x_0+ih$, $(i=0, \cdots, n)$. These results are used here to invert the Vandermonde matrix

$$V(x_1, \cdots, x_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix},$$

where the x_i are distinct, different from zero, but otherwise arbitrary.

The paper is in three parts. In the first we outline the results from [1] required later. In the second, these formulas are applied to the special Vandermonde matrix

$$V_{n+1}(x_0, h) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_0+h & \cdots & x_0+nh \\ \cdot & \cdot & \cdots & \cdot \\ x_0^n & (x_0+h)^n & & (x_0+nh)^n \end{bmatrix};$$

and the elements of $V_{n+1}^{-1}(x_0, h)$ are obtained in terms of Stirling numbers. Finally, the methods of [1] are extended in such a way that the elements of $V^{-1}(x_1, \cdots, x_n)$ can be expressed in terms of the elementary symmetric functions of the x's. This last result is offered as an alternative to the derivation one would obtain from the classical formula for the values of Vandermonde determinants with missing powers ([2], p. 99).

2. Preliminary results. Let $y=f(x)$ be a polynomial of degree n, and $y_i=f(x_i)$, $(i=0, 1, \cdots, n)$, where $x_i=x_0+ih$. It was shown in [1] that if we write

$$h^k f^{(k)}(x) = \sum_{i=0}^{n} A_{mi}^k y_i, \tag{1}$$

then

$$A_{mi}^k = \sum_{j=k}^{n} \frac{(-1)^{i+j}\binom{j}{i} C_{mj}^k}{j!}, \tag{2}$$

where

* Now at Alabama Polytechnic Institute and University of Wisconsin-Milwaukee, respectively.

$$\binom{j}{i} = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ if } i > j, \end{cases}$$

$$C_{mj}^{k} = \sum_{r=k}^{j} p_k(r)\ S_j^r m^{r-k}, \tag{3}$$

m is any real number, and $x = x_0 + mh$. In the above, $p_k(r)$ denotes the factorial polynomial of degree k, and the S_k^j are the Stirling numbers of the first kind. Thus, we have $p_k(x) = \sum_{j=1}^{k} S_k^j x^j$. A wide variety of classical numerical differentiation formulas are special cases of (1).

It was shown further that these results enable one to invert the Vandermonde matrix

$$M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -m & 1-m & \cdots & n-m \\ \cdot & \cdot & \cdots & \cdot \\ (-m)^n & (1-m)^n & \cdots & (n-m)^n \end{bmatrix}.$$

If we write $M^{-1}(m) = \{a_{\lambda,\mu}^{m}\}$, $(\lambda, \mu = 1, \cdots, n+1)$, then

$$a_{\lambda,\mu}^{m} = \frac{A_{m,\lambda-1}^{\mu-1}}{(\mu-1)!}, \tag{4}$$

where $A_{m,\lambda-1}^{\mu-1}$ is given by (2).

3. Vandermonde Matrices for equally spaced points. It is easy to show that

$$\begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h & 0 \cdots 0 \\ \cdot & \cdot & \cdots \\ 0 & 0 & 0 \cdots h^n \end{bmatrix} M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -mh & -mh+h & \cdots & -mh+nh \\ \cdot & \cdot & \cdots & \cdot \\ (-mh)^n & (-mh+h)^n & \cdots & (-mh+nh)^n \end{bmatrix}$$

If we write $x_0 = -mh$, it follows that

$$V_{n+1}(x_0, h) \equiv \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_0+h & \cdots & x_0+nh \\ \cdot & \cdot & \cdots & \cdot \\ x_0^n & (x_0+h)^n & \cdots & (x_0+nh)^n \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h & 0 \cdots 0 \\ \cdot & \cdot & \cdots \\ 0 & 0 & 0 \cdots h^n \end{bmatrix} M(-x_0/h),$$

and so

$$V_{n+1}^{-1}(x_0, h) = M^{-1}(-x_0/h)\begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h^{-1} & 0 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots h^{-n} \end{bmatrix}.$$

Thus, by (4) the element in the λth row and μth column of $V_{n+1}^{-1}(x_0, h)$ is precisely

$$v_{\lambda\mu} = \frac{1}{h^{\mu-1}(\mu - 1)!} A^{\mu-1}_{-x_0/h, \lambda-1}.$$

It is interesting to note that even though $V_{n+1}^{-1}(1, 1)$ can be obtained directly from the above, a slight modification enables one to express the elements of this inverse very compactly. For simplicity of notation, we apply the method to obtain $V_n^{-1}(1, 1)$.

Let us write

$$M(0) = \begin{bmatrix} 1 & Q_{1n} \\ R_{n1} & P_{nn} \end{bmatrix}, \qquad M^{-1}(0) = \begin{bmatrix} b_{11} & S_{1n} \\ T_{n1} & U_{nn} \end{bmatrix},$$

where $Q_{1n} = (1, \cdots, 1)$, R_{n1} is a column vector of zeros, and

$$P_{nn} = \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 4 & \cdots & n^2 \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 2^n & \cdots & n^n \end{bmatrix}.$$

It suffices to invert the matrix P_{nn}, since its columns are scalar multiples of those of $V_n(1, 1)$. Now

$$M(0)M^{-1}(0) = \begin{bmatrix} b_{11} + Q_{1n}T_{n1} & S_{1n} + Q_{1n}U_{nn} \\ R_{n1}b_{11} + P_{nn}T_{n1} & R_{n1}S_{1n} + P_{nn}U_{nn} \end{bmatrix} = I,$$

and so $I_n = R_{n1}S_{1n} + P_{nn}U_{nn} = P_{nn}U_{nn}$. Hence $U_{nn} = P_{nn}^{-1}$. Denote the elements of U_{nn} by μ_{ik}. Since b_{11} contains only a single element, it follows that $\mu_{ik} = a^0_{i+1,k+1}$, and in turn, by (4), $\mu_{ik} = A^k_{0i}/k!$. We have from (2) that

$$A^k_{0,i} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} C^k_{0j}}{j!} \qquad (i, k = 1, \cdots, n).$$

From (3) $C^k_{0j} = p_k(k)S^k_j = k!S^k_j$. Finally, we have

$$A_{0i}^{k} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} k! S_j^k}{j!},$$

and so, by (4),

$$\mu_{ik} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} S_j^k}{j!}. \tag{5}$$

Since the kth column of V_{nn} (1, 1) is k^{-1} times the kth column of P_{nn}, $(k=1, \cdots, n)$, it follows that $V_n^{-1}(1, 1) = \{i\mu_{ik}\}$, $(i, k=1, \cdots, n)$, where μ_{ik} is given by (5). As an illustration, the following inverses are given:

$$P_{44}^{-1} = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -3 & \frac{19}{4} & -2 & \frac{1}{4} \\ \frac{4}{3} & -\frac{7}{3} & \frac{7}{6} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{11}{24} & -\frac{1}{4} & \frac{1}{24} \end{bmatrix}, \quad V_4^{-1}(1, 1) = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -6 & \frac{19}{2} & -4 & \frac{1}{2} \\ 4 & -7 & \frac{7}{2} & -\frac{1}{2} \\ -1 & \frac{11}{6} & -1 & \frac{1}{6} \end{bmatrix}.$$

These inverses may be checked by multiplication with the corresponding direct matrix. P_{44}^{-1} was obtained directly, by use of (5).

4. The inverse of the general Vandermonde matrix. Let $x_0=0, x_1, \cdots, x_n$ be the given distinct numbers. The polynomial $y=y(x)$ of degree n assuming $n+1$ arbitrary values $y_i=y(x_i)$, $i=0, \cdots, n$, can be written, by Lagrange's interpolation formula, as

$$y = \sum_{i=0}^{n} A_{x,i} y_i. \tag{6}$$

If we write the kth derivative of $y(x)$ as

$$y^{(k)} = \sum_{i=0}^{n} A_{xi}^{k} y_i \qquad (k = 1, \cdots, n), \tag{7}$$

where

$$A_{x,i}^{k} = \frac{d^k}{dx^k} A_{x,i},$$

and set $x=x_0=0$, we have

$$y_0^{(k)} = \sum_{i=0}^{n} A_{0,i}^{k} y_i. \tag{8}$$

In the following, we obtain explicit expressions for the $A_{0,i}^{k}$, and then show that the $A_{0,i}^{k}$ satisfy systems of linear equations having the same coefficient matrix. Thus, we obtain the inverse of this matrix and, in turn, the inverse of $V(x_1, \cdots, x_n)$ in terms of the $A_{0,i}^{k}$.

The functions A_{xi}, as given by the Lagrange interpolation formula, are

$$\begin{aligned} A_{xi} &= \frac{1}{p_n'{}_{+1}(x_i)} \cdot \frac{p_{n+1}(x)}{(x - x_i)} \\ &= \frac{1}{p_n'{}_{+1}(x_i)} (x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n); \end{aligned}$$

and so

$$\begin{aligned} A_{xi} = \frac{1}{p_n'{}_{+1}(x_i)} [x^n &- \sigma_{1,n-1}^{i} x^{n-1} + \sigma_{2,n-1}^{i} x^{n-2} - \cdots \\ &+ (-1)^{n-2} \sigma_{n-2,n-1}^{i} x^2 + (-1)^{n-1} \sigma_{n-1,n-1}^{i} x], \end{aligned}$$

where $\sigma_{j,n-1}^{i}$ is the sum of all products of j of the numbers $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ without permutations or repetitions ($\sigma_{0,n-1}^{i} \equiv 1$). If we differentiate the above k times, set $x = 0$, and notice that

$$p'_{n+1}(x_i) = \prod_{j=0}^{n}{}' (x_i - x_j),$$

where the dash indicates that $i \neq j$, we obtain*

$$A_{0i}^{k} = \frac{(-1)^{n-k} \cdot k!}{\prod_{j=0}^{n}{}' (x_i - x_j)} \cdot \sigma_{n-k,n-1}^{i}. \tag{9}$$

In order to exhibit the linear systems mentioned above, we expand $y(x)$ about $x_0 = 0$, and substitute $x = x_i$ to get

$$y_i = \sum_{\mu=0}^{n} \frac{1}{\mu!} y_0^{(\mu)} x_i^{\mu}.$$

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* More generally, if we do not substitute $x = 0$ after the above differentiation, and substitute the result in (7), we obtain

$$y^{(k)}(x) = \sum_{i=0}^{n} y_i \sum_{j=k}^{n} \frac{(-1)^{n-j} \sigma_{n-j,n-1}^{i}}{p_{n+1}(x_i)} j(j-1) \cdots (j-k+1) x^{j-k},$$

which is a formula for an arbitrary derivative of $y(x)$ at an arbitrary x.

Inserting this into (8) and rearranging, we get

$$y_0^{(k)} = \sum_{\mu=0}^{n} \frac{1}{\mu!} y_0^{(\mu)} \sum_{i=0}^{n} x_i^{\mu} A_{0i}^{k} \qquad (k = 1, \cdots, n).$$

Since these are identities in $y_0^{(k)}$, we must have

$$\sum_{i=0}^{n} x_i^{\mu} A_{0i}^{k} = \delta_{\mu k} k! \qquad (\mu = 0, 1, \cdots, n),$$

where $\delta_{\mu k}$ is the Kronecker delta. Since $x_0 = 0$, it follows that

$$\sum_{i=1}^{n} x_i^{\mu} A_{0i}^{k} = \delta_{\mu k} \cdot k! \qquad (\mu = 1, \cdots, n).$$

For each k ($k = 1, \cdots, n$), this is a system of n linear equations in the unknowns $A_{01}^{k}, A_{02}^{k}, \cdots, A_{0n}^{k}$. These n systems can be combined into the matrix equation

$$\left\{x_k^i\right\} \cdot \left\{A_{0i}^{k}\right\} = \begin{bmatrix} 1! & 0 & 0 \cdots 0 \\ 0 & 2! & 0 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots n! \end{bmatrix}$$

where A_{0i}^{k} is the element of $\left\{A_{0i}^{k}\right\}$ in the ith row and kth column, and similarly for the element x_k^i of $\left\{x_k^i\right\}$. (In x_k^i the i denotes an actual exponent.) Denoting the right member of the above equation by $D(n)$, we may write $\left\{A_{0i}^{k}\right\} = \left\{x_k^i\right\}^{-1} \cdot D(n)$. Thus, the element $b_{\lambda\mu}$ in the λth row and μth column of $\left\{x_k^i\right\}^{-1}$ ($\lambda, \mu = 1, \cdots, n$) is given by

$$b_{\lambda\mu} = \frac{1}{\mu!} A_{0\lambda}^{\mu}, \tag{10}$$

where the $A_{0\lambda}^{\mu}$ may be obtained from (9).

Finally, if $V^{-1}(x_1, \cdots, x_n) \equiv \left\{v_{\lambda\mu}\right\}$, a method similar to that used in Section 3 yields $v_{\lambda\mu} = x_\lambda b_{\lambda\mu}$, ($\lambda, \mu = 1, \cdots, n$), which, together with (10), gives an explicit representation for the inverse of the general Vandermonde matrix.

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ON THE CURRICULUM FOR PROSPECTIVE HIGH SCHOOL TEACHERS

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Within the framework of a National Science Foundation Institute held at Stanford University in the present academic year 1957–58 I am conducting two classes for high school teachers:

(1) *Seminar in Problem Solving*: 2 hours in the Fall and Winter Quarters, 3 hours in the Spring Quarter.

(2) *From Elementary Mathematics to the Calculus to Scientific Method* (the official title is a little different): 4 hours in the first two quarters, 3 hours in the last quarter.

In the last Summer Quarter, I had essentially the same classes, one for General Electric Fellows, the other for Shell Merit Fellows, although in a much abridged form; the contents and the form of presentation have been developed in regular university lectures given at Stanford since 1942. I think that these classes fill an important gap in the curriculum for prospective high school teachers and, therefore, I take the liberty to say a few words about them.

(1) The aim is to give the teachers experience in genuine, nonroutine mathematical work which, at this level, cannot be "research" but just "problem solving." (To my knowledge, neither the departments of mathematics nor the schools of education offer the teachers such experience—if there are exceptions, they are certainly rare.) Problems are solved in class discussion led by the instructor. The problems are not always easy, but they are on high school level, or only slightly above it. In the first phase, the problems are grouped according to subject matter. Geometric constructions with ruler and compasses; setting up equations; binomial coefficients and arithmetic series of higher order—here are three subjects which I found particularly appropriate. In the later phases of the seminar, the problems are grouped according to method to illustrate general ideas of problem solving. At the end, the participants in the class should be given opportunity to take the place of the instructor and lead the discussion.*

(2) This is a short course in analytic geometry and calculus, including a very sketchy last chapter on differential equations. Yet the course is pretty different from the usual: connections with elementary mathematics are emphasized at the beginning, applications to science are discussed at the end, general methodical ideas are stressed all along. Pat solutions are avoided, heuristic reasoning and historical sources are often in the foreground.

* The underlying ideas have been expressed in my books *How to Solve It* and *Mathematics and Plausible Reasoning*; the difficulty is to adapt these ideas to a specific level. I am working on a book presenting the materials of the seminar; the difficulty is to make the book usable by a not specifically prepared instructor.

"Mathematics as a Language" is the title of the first chapter. The "language of algebraic formulas" is contrasted with the "language of geometric figures" which speaks to us in graphs and diagrams. Analytic geometry is introduced as a "dictionary of two languages."

"The Beginnings of the Integral Calculus" is the next chapter; problems are solved by Archimedes' mechanical method, by Cavalieri's principle, and by the "method of exhaustion" before Leibnitz's notation and the usual scheme is introduced.

There is no space here to sketch the rest of the contents, but something should be said about the reasons which led to the outlined choice of subjects. I wish to present these remarks under the guise of informal advice (such as I am inclined to give in my classes).

1. There is one infallible teaching method: you will infallibly succeed in boring your audience with your subject if you are bored with it yourself. Hence the first commandment for teachers: *Be interested in your subject.*

2. No amount of courses in teaching methods will enable you to explain understandably a point that you do not understand yourself. Hence the second commandment for teachers: *Know your subject.*

3. Our knowledge about any subject consists of "information" and of "know-how." In mathematics "know-how" is the ability to solve problems and it is much more important than mere possession of information. You have to show your students how to solve problems—can you show it if you don't know it? Hence a special commandment for mathematics teachers: *Acquire, and keep up, some aptitude for problem solving.*

4. You may be obliged to discuss many problems which have little lasting interest in themselves. Yet you should use them to develop your students (and yourself). Therefore: *Look out for such features of the problem at hand as may be useful in solving the problems to come.*

5. The number of such features is unlimited. Here is a table (see Table 1) of some which seem to me the most important in acquiring good habits of mind—these are the points that, in my opinion, deserve the most attention, and should be stressed at each reasonable opportunity, in a high school mathematics class. More space than here available and, in the first place, many examples were needed to explain the points collected in Table 1 satisfactorily.† Yet here are a few hints.

6. In any problem something must be *unknown*—otherwise there would be nothing to do, nothing to look for. Yet the unknown must be specified somehow —and it cannot be specified unless something is known or given. The essential

† See the books quoted in footnote *.

Unknown Data Condition
Generalization Specialization
Analogy
Strict Reasoning
Reasonable Guessing
Language
of
Diagrams Formulas

TABLE 1. SOME GENERAL AIMS OF THE HIGH SCHOOL CURRICULUM

circumstances which link the unknown to the given things, or *data*, are enumerated by the *condition*. We cannot hope to solve a worthwhile problem unless we know, and know very well, what is the unknown, what are the data, and what is the condition. The student should acquire the habit of paying proper attention to unknown, data, and condition—a habit more important for his mental development and his later studies and professional work than the knowledge of any particular mathematical fact.

7. Generalities without interesting particular cases are of little value and so are particular facts without some hope of *generalization*. What is really valuable is ready ascent from particular facts to generalizations and ready descent from generalizations to particular facts.

8. *Analogy* is the great guide of invention. The best may be missing from a mathematics curriculum in which the student never meets an impressive example of discovery by analogy.

9. It has often been said that mathematics is a good school of *strict reasoning*. In fact more is true: strict (conclusive, "logical" "deductive") reasoning is essentially confined to mathematics; it deals only with objects lifted to the logical-mathematical level. What most students need in this respect is only so much contact with strict proofs that they get into the habit of clearly distinguishing between conclusive and inconclusive reasoning, between a proof and a guess.

Yet the student should also learn to distinguish between guesses and guesses, between a good guess and a bad guess, between a good guess and a better guess. Now (this is not so well known and, therefore, it must be said with great emphasis) mathematics is also a good school of *reasonable guessing*, of plausible (inconclusive, "inductive") reasoning. There is no space here to develop the

arguments for this assertion.‡ Yet let me give a little hint for classroom use: If there is a reasonable opportunity, *start a new problem or a new subject by letting your students guess*. Having guessed, they commit themselves and have to follow developments to see whether their guess comes true.

10. What is the aim of the high school mathematics curriculum? I do not intend to compete with all the big and beautiful words which allegedly answer this question. Yet let us put the question more clearly: By what standard should we judge the success of a high school curriculum in mathematics? By essentially the same standard, I say, as we would judge the curriculum in French: by the facility that the students acquire in using the language—since mathematics is essentially a language. After graduation, your former student will go to college or into some profession. In the one place as in the other, he may face a problem capable of mathematical treatment. If he can reduce it to a neat computation, or set up an equation for it, or express it by a diagram, the result of your teaching is excellent. If, without being able to produce them, he can read graphs and formulas, and appreciate them, too, the result is still very good. If, however, he cannot read graphs, cannot read formulas, and does not care for them a bit, the result is poor.

‡ See especially the second work quoted in footnote*.

CORRECTION

Raphael M. Robinson, *The converse of Fermat's theorem*, this MONTHLY, vol. 64, 1957, pp. 703–710. The formula in Theorem 10, page 709 should be "$a^{(N-1)/2} \equiv -1 \pmod{N}$." The error was made by the editorial staff, not by the author.

MATHEMATICAL NOTES

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ON THE SERIES $\sum 1/p$

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For the fact that the sum of the reciprocals of the primes diverges, Euler was the first to give a (somewhat incomplete) proof [1]. Variations of Euler's proofs have been given by numerous mathematicians [2] and a number of

essentially different proofs have been given recently by Erdös [3] and Bellman [4]. The purpose of this note is to add one more to this number.

We begin with a lemma concerning subseries of the harmonic series. Let $\{p\} = p_1 < p_2 < \cdots$ be a sequence of positive integers and let

$$\pi(x) = \sum_{p \le x} 1 \quad \text{and} \quad R(x) = \sum_{p \le x} 1/p.$$

LEMMA. *If* $\lim_{x\to\infty} R(x)$ *exists, then* $\lim_{x\to\infty} \pi(x)/x = 0$.

Proof. We have

$$\pi(x) = 1\{R(1) - R(0)\} + 2\{R(2) - R(1)\} + \cdots + x\{R(x) - R(x-1)\}.$$

Hence

$$\pi(x)/x = R(x) - [\{R(0) + R(1) + \cdots + R(x-1)\}/x]. \tag{1}$$

If $\lim_{x\to\infty} R(x)$ exists, the square bracket in (1) approaches this limit, so that $\pi(x)/x \to 0$ as required.

In what follows, we let the p's be the consecutive primes and assume that $\lim_{x\to\infty} R(x)$ exists. This will lead to a contradiction and thus prove our theorem.

By our assumption and the lemma, there exist integers $n > 1$ and m such that

$$\sum_{p>n} 1/p < 1/2, \tag{2}$$

$$\pi(n!m)/m < 1/2. \tag{3}$$

With such n and m form the m integers $T_i = in! - 1$, $i = 1, \cdots, m$. All prime factors of the T_i exceed n and, if $p \mid T_i$ and $p \mid T_j$, then $p \mid (T_i - T_j)$ so that $p \mid (i-j)$. Hence a prime p is a divisor of at most $(m/p)+1$ of the T_i. Since each T is divisible by at least one prime p in the range $n!m > p > n$, we deduce

$$\sum_{n!m>p>n} \left(\frac{m}{p} + 1\right) \ge m. \tag{4}$$

From (4) we have

$$\sum_{p>n} \frac{1}{p} + \frac{\pi(n!m)}{m} \ge 1. \tag{5}$$

But now (2), (3), and (5) yield the desired contradiction.

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ON THE MINIMIZATION OF MATRIX NORMS

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A well-known inequality, due to Schur [1], states that *if A is a complex $n\times n$ matrix with euclidean norm $\|A\|$ and characteristic roots $\omega_1, \cdots, \omega_n$, then*

$$\|A\|^2 \geqq \sum_{i=1}^{n} |\omega_i|^2. \tag{1}$$

Equality occurs in (1) *if and only if A is normal.* Commenting on this result, Wedderburn [2] observed: "Since replacing a matrix by a similar one corresponds to a change of coordinates when the matrix is regarded as a linear transformation, it follows from Schur's work that, when the elementary divisors of A are simple, $\|A\|$ has its minimum value when A is represented as a diagonal matrix, that is, in its normal form; and it seems probable that the normal form also gives the minimum value even if the elementary divisors are not simple." Wedderburn thus conjectured that, for a fixed matrix A, $\|S^{-1}AS\|$ is minimal when $S^{-1}AS$ is the classical canonical form of A. This would imply, in particular, that, for a nondiagonable matrix A and any nonsingular matrix S,

$$\|S^{-1}AS\|^2 \geqq \sum_{i=1}^{n} |\omega_i|^2 + 1.$$

We shall demonstrate that Wedderburn's conjecture is incorrect by establishing the following result.

THEOREM. *Let A be a complex $n\times n$ matrix with characteristic roots $\omega_1, \cdots, \omega_n$. Then*

$$\inf \|S^{-1}AS\|^2 = \sum_{i=1}^{n} |\omega_i|^2, \tag{2}$$

where the lower bound is taken with respect to all nonsingular matrices S. Furthermore, the lower bound is attained if and only if A is diagonable.

Denote by T_1 a matrix such that $T_1^{-1}AT_1$ is triangular, say

$$T_1^{-1}AT_1 = \begin{Vmatrix} \omega_1 & b_{12} & \cdots & b_{1n} \\ 0 & \omega_2 & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \omega_n \end{Vmatrix}.$$

Let δ be an arbitrary positive number $\leqq 1$, and write $T_2 = \operatorname{diag}(1, \delta, \cdots, \delta^{n-1})$, $S_0 = T_1T_2$. Then

$$\|S_0^{-1}AS_0\|^2 = \sum_{i=1}^{n} |\omega_i|^2 + \sum_{1\leq i<k\leq n} |b_{ik}|^2\delta^{2(k-i)}$$

$$\leqq \sum_{i=1}^{n} |\omega_i|^2 + \frac{1}{2} n(n-1)b^2\delta^2,$$

where $b = \max_{i,k} |b_{ik}|$. The relation (2) now follows since, by (1),

$$\|S^{-1}AS\|^2 \geqq \sum_{i=1}^{n} |\omega_i|^2$$

for every nonsingular matrix S. Furthermore, if A is diagonable, then the lower bound of $\|S^{-1}AS\|$ is clearly attained. If, on the other hand, the bound is attained so that, for some S,

$$\|S^{-1}AS\|^2 = \sum_{i=1}^{n} |\omega_i|^2,$$

then $S^{-1}AS$ is normal and A is diagonable.

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MATRIX NIM

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The original game of Nim is the game for which a position consists of a finite set of nonnegative integers, called piles. A legal move consists of lowering the size of one and only one pile. Two players take turns moving until all of the piles are reduced to zero, in which case the player that makes the last move wins. For games such as this, where players take turns making the same type of moves (which I call termination games), it is easily shown that positions may be divided into two uniquely determined classes, called safe and unsafe ([3], Th. 9), such that a player may force a win from an opponent whose turn it is if and only if the position is safe.

In 1902, C. L. Bouton [1] showed that a position in Nim is safe if and only if expressing the pile numbers as binary numbers, each power of two is represented an even number of times. For instance, in 3-pile Nim, a safe position is (49, 36, 21). Since $49=32+16+1$, $36=32+4$ and $21=16+4+1$, we have 32 represented twice, 16 twice, 8 no times, 4 twice, 2 no times and 1 twice.

In 1910, E. H. Moore [2] defined a position for "Nim$_k$" (k a positive integer) as the same as for ordinary Nim (which is "Nim$_1$"), and a move for "Nim$_k$" would consist of lowering any number of piles, not to exceed k. He showed that, in this game, a position is safe if and only if, when the pile numbers are expressed as binary numbers, each power of two is represented zero times modulo $k+1$.

* Work done under the auspices of the AEC.

My paper on termination games [3] shows that for combining termination games in a certain way, the game of Nim enters in a fundamental manner. In that paper, I also generalize "Nim$_k$" to positions consisting of ordinal numbers.

In this paper, still another generalization of Nim is discussed. Let n be a positive integer. Then let a position for Nim^n (superscripts, instead of subscripts, are used to avoid confusion with Moore's game) be a rectangular array of nonnegative integers, called piles, such that there are n and only n columns and a finite number of rows. Let a legal move in Nim^n consist of either (1) Lowering the sizes of any nonempty set of piles, provided they are all in the same row, or (2) Lowering the sizes of any nonempty set of piles, provided they are all in the same $n-1$ columns, or equivalently, provided that at least one column is left untouched.

A game is terminated when all of the piles are reduced to zero, and whoever terminates a game wins it. Notice that Nim^1 is the ordinary game of Nim.

THEOREM. *A position in Nim^n is safe if and only if the following two conditions are both met*:

(a) *The sum of the elements in any column is equal to the sum of the elements in any other column.*

(b) *For each row, consider the smallest pile in it. Then these minimum pile sizes, one from each row, constitute a safe position in ordinary Nim.*

Proof. That a dichotomic labeling of positions into safe and unsafe satisfies the above win-and-lose condition, it is necessary and sufficient that it satisfy the following two conditions ([3], Par. 4 and Th. 9).

I. Given a safe position, any move leads to an unsafe position.

II. Given an unsafe position, there exists a move which leads to a safe position.

Let us first show that Condition I is satisfied. Given a safe position, a move would destroy (a), the equality of column sums, unless it was of Type (1) such that each element of the row was reduced by the same amount. However, the smallest size for the row would be lowered so that Condition (b) would no longer be satisfied.

To show Condition II, consider any unsafe position. If (b) is not satisfied, choose a row R for which one may lower its minimum size to a size M to satisfy (b). If (b) is satisfied (so (a) is not), choose any row R and let M be its minimum size. Choose a column C such that the sum of the elements in C is less than or equal to any other column sum. Now the situation breaks up into the following two cases:

Case A. If any element of R which is not in C is changed to M, then the new sum for the column containing this element will be less than the sum for column C.

Case B. There exists some element of R, which is not in C, such that if its size is reduced to (or left at) M, then the new sum for this column will be greater than or equal to the sum for C.

For Case A, one may move to a safe position by making a move of Type (1). For each element of R, consider the column sum for its column were this element to be made equal to M. Choose the element which gives the maximum such column sum and make it equal to M. Now for each of the other elements of R, put its size at the value which will make its column sum equal to the column sum of the previous sentence. Then the minimum for R will be M.

For Case B, one may move to a safe position by making a move of Type (2). Leave column C unaltered. Put one of the elements in R, not in C, to M so that the resulting column sum is at least as big as the sum for C. Consider each element as being the sum of the minimum for its row and a nonnegative integer, which we will call its excess. Then the sum of the excesses for each column is greater than or equal to the sum of the excesses for C. Whenever this sum is greater than the sum for C, lower the excesses until equality is achieved. Then Condition (b) as well as Condition (a) will be satisfied and we will have obtained a safe position.

An interesting, as yet unsolved, problem is to analyze the game of Matrix Nim where instead of permitting a move of Type (2), one may lower the sizes of any nonempty set of piles, provided they are all in the same column.

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A GENERAL THEORY FOR LINEAR SYSTEMS*

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A singular linear transformation L does not have an inverse. Nevertheless, it is always possible to find a partial or pseudo-inverse M of L, that is, a linear transformation M such that $LML=L$, [1] (Proposition 2, p. 179). If L^{-1} exists, of course, then L^{-1} is a pseudo-inverse, for $LL^{-1}L=L$.

From the equation $LML=L$ one can deduce an important application of the pseudo-inverse. Suppose that the linear equation $Lx=y_0$, in which y_0 denotes a

* This paper was written while the author was an Oak Ridge Institute of Nuclear Studies Fellow and was presented at the meeting of the Louisiana-Mississippi Section of the Mathematical Association of America, February 15, 1957.

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given vector, is known to be consistent; that is, there exists x_0 such that $Lx_0=y_0$. Then since

$$LMy_0 = LMLx_0 = Lx_0 = y_0,$$

it follows that My_0 is a solution of the given linear equation.

A pseudo-inverse is also useful in solving the homogeneous equation $Lx=0$. Since L is singular, $I-ML\neq 0$. If z is in the domain of L, then

$$L(I - ML)z = Lz - LMLz = 0;$$

in other words, $x=(I-ML)z$ is a solution of the homogeneous equation $Lx=0$. In fact, $I-ML$ projects X onto the null space, $N(L)$, of L.

In the finite-dimensional case L is a matrix, and an explicit calculation is presented here for a pseudo-inverse M of L. This procedure is suitable for machine programming and leads to an M with the desirable feature that the nonzero columns of $I-ML$ form a basis for the null space of L.

The process makes use of the Hermite canonical form of an $n\times n$ matrix.

1.1. DEFINITION. *A matrix $H=(h_{ij})$ is said to be in Hermite canonical form provided*

(1) $h_{ij}=0$, *whenever* $i>j$;

(2) $h_{ii}=0$ *or* 1;

(3) *if* $h_{ii}=0$, *then* $h_{ik}=0$, $(k=1,\cdots,n)$; *and*

(4) *if* $h_{ii}=1$, *then* $h_{ji}=0$, $(j=1,\cdots,n, j\neq i)$.

1.2. THEOREM. *For a given $n\times n$ matrix L, there exists a nonsingular matrix M which is a product of elementary matrices such that $ML=H$ is in Hermite canonical form.*

A proof of this theorem is given in many of the standard texts (see, for example [2], Theorem 18, p. 35).

1.3. LEMMA. *If H is in Hermite canonical form, then $H^2=H$.*

Proof. Let a_{ik} denote an arbitrary element of H^2. By 1.1, (1), $a_{ik}=\sum_{j=1}^{n} h_{ij}h_{jk}$ vanishes if $i>k$ and reduces, if $i\leqq k$, to $\sum_{j=i}^{k} h_{ij}h_{jk}$. By 1.1, (2), $h_{ii}=0$ or 1; and if $h_{ii}=0$, then by 1.1, (3), $h_{ij}=0$ for all j and thus $a_{ik}=0=h_{ik}$. If $h_{ii}=1$, then

$$a_{ik} = h_{ik} + \sum_{j=i+1}^{k} h_{ij}h_{jk} .$$

Now whenever $h_{ij}\neq 0$ for some $j>i$, it follows from 1.1, (4) that $h_{jj}=0$, and hence from (3) that $h_{jm}=0$ for all m and in particular for $m=k$. Hence, in any case, $a_{ik}=h_{ik}$ and $H^2=H$.

1.4. THEOREM. *If M is the product of the elementary matrices which reduce L to H, then M is a nonsingular pseudo-inverse of L.*

Proof. By hypothesis $ML=H$, and since M is a product of elementary matrices, M^{-1} exists; therefore, $L=M^{-1}H$. Hence

$$LML = M^{-1}HMM^{-1}H = M^{-1}H^2 = M^{-1}H = L.$$

M can be computed by performing on the identity matrix the elementary row operations which reduce L to H. If, however, one is interested only in the solution of $Lx=y_0$ it is not necessary to compute M. To solve the problem one proceeds in the following manner. Form the $n\times(n+1)$ matrix $L^+=(L, y_0)$, and then perform the elementary row operations on L^+ which reduce L to H. The resulting matrix ML^+ is of the form (H, My_0), where H is of the form

$$\begin{bmatrix} 1 & a & 0 & b & d \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the only nonzero rows of H are those with a 1 in the diagonal, it follows that the rank of L, which equals the rank of H, is given by the number of 1's along the diagonal. Moreover, any column which has a 1 in the diagonal is a unit vector by 1.1, (4). Since the $(n+1)$st column of L^+ becomes My_0 as L is reduced to H and since L and M are full inverses of each other on the range of L and the range of ML, it follows that the equation $Lx=y_0$ is consistent if and only if My_0 has nonzero components only in the rows in which H has 1's. In this case, My_0 is a particular solution of $Lx=y_0$. If $H=ML$ has r 1's down the diagonal, then ML contains r unit column vectors and hence $I-ML=I-H$ contains exactly $n-r$ nonzero columns. Since $(I-ML)u=u$ for arbitrary $u\in N(L)$, it follows that these columns of $I-ML$ span $N(L)$; and since $N(L)$ is of dimension $n-r$, it even follows that they are linearly independent. Thus the complete solution of $Lx=y_0$ is given by

$$x = My_0 + (I - ML)D,$$

in which D is an arbitrary diagonal matrix containing only $n-r$ effective parameters since $I-ML$ has only $n-r$ nonzero columns.

References

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ON AN EXISTENCE THEOREM FOR COMPLEX-VALUED DIFFERENTIAL EQUATIONS*

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The purpose of this note is to provide an alternative proof of a theorem stated in the recent—and excellent—textbook on ordinary differential equations by Coddington and Levinson [1]. This theorem (Th. 8.4, p. 36) is stated for a system of equations involving several complex parameters. We shall state and prove the theorem for a single equation, namely

$$w' = f(t, w, \lambda), \tag{1}$$

involving only a single complex parameter λ. In addition, we strengthen slightly both the hypotheses and the conclusion in stating

THEOREM I. *Let $f(t, w, \lambda)$, f_w, f_λ be continuous on a domain*

$$D_\lambda: t \in I_t,\ w \in D,\ |\lambda - \bar{\lambda}| < c.$$

For fixed $\lambda = \bar{\lambda}$, let $w(t)$ be a solution of (1) on a subinterval I, $a \leqq t \leqq b$, of I_t. Then there exists $\delta > 0$ such that for any point (t^0, w^0, λ^0) of U_δ, where

$$U_\delta: a < t^0 < b,\ |w^0 - w(t^0)| + |\lambda^0 - \bar{\lambda}| < \delta,$$

there exists a unique solution $W(t; t^0, w^0, \lambda^0)$ of (1) on I, passing through the point (t^0, w^0). Moreover, $W \in C^1$ on the domain

$$V_\delta: a < t < b,\quad (t^0, w^0, \lambda^0) \in U_\delta,$$

and for each fixed (t, t^0), W is an analytic function of $(w^0, \lambda^0) \in U_\delta$.

In this theorem, D is a domain of the complex w-plane, the function $f = f_1 + if_2$ is complex-valued, and $\lambda = \lambda_1 + i\lambda_2$ is a complex parameter. For fixed λ, a solution of (1) is a complex-valued function $w(t) = w_1(t) + iw_2(t)$ defined on an interval I of the t-axis, of class C^1 on I (*i.e.*, $w_1(t)$, $w_2(t) \in C^1$), and satisfying the equation (1) on this interval (so that $w(t) \in D$ for $t \in I$).

In their book the authors suggest that the proof may be carried out by the method of successive approximations. This is indeed true, and as the authors remark, the proof is very similar to that of a corresponding theorem involving only real functions, the analyticity following from the uniformity of convergence. The proof we intend to give, on the other hand, makes use of the corresponding theorem for real functions—or rather, of the corresponding theorem involving a system (of two equations) involving two (real) parameters ([1], Th. 7.5, p. 30). Part of the interest in this alternative proof is due to the fact that it brings to the fore the role played by the theory of functions of two complex variables, in particular the Cauchy-Riemann equations.

For completeness, we restate the form of Theorem 7.5 [1] which we intend to use, as

* This note was written while the author held a grant from the National Research Council of Canada.

THEOREM II. *Consider the (real) system of differential equations*

$$w_1' = f_1(t, w_1, w_2; \lambda_1, \lambda_2), \qquad w_2' = f_2(t, w_1, w_2; \lambda_1, \lambda_2),$$

where the λ_i are real parameters. Let D be a domain of (t, w_1, w_2) space, and let I_λ be defined by $|\lambda_1-\bar{\lambda}_1|+|\lambda_2-\bar{\lambda}_2|<c$. By D_λ we mean the domain of $(t, w_1, w_2, \lambda_1, \lambda_2)$ space defined by

$$D_\lambda\colon (t, w_1, w_2) \in D, \qquad (\lambda_1, \lambda_2) \in I_\lambda.$$

Suppose that f_i, $\partial f_i/\partial w_j$, $\partial f_i/\partial \lambda_j$ $(i, j=1, 2)$ are all continuous on D_λ. For the fixed values $\lambda_i=\bar{\lambda}_i$, let $(w_1(t), w_2(t))$ be a solution of (2) on an interval $a\leqq t\leqq b$. Then there exists $\delta>0$ such that for any point $(t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)\in U_\delta$, where

$$U_\delta\colon a<t^0<b,\ |w_1^0-w_1(t^0)|+|w_2^0-w_2(t^0)|+|\lambda_1^0-\bar{\lambda}_1|+|\lambda_2^0-\bar{\lambda}_2|<\delta,$$

there exists a unique solution $W_i(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)$ $(i=1, 2)$ of (2) on $a\leqq t\leqq b$ passing through the point (t^0, w_1^0, w_2^0) of D. Moreover, $W_i\in C^1$ on the domain

$$V_\delta\colon a<t<b, \qquad (t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)\in U_\delta. \tag{3}$$

In order to prove Theorem I we first note ([2], Ch. IX) that the hypotheses on f imply that the eight partial derivatives $\partial f_i/\partial w_j$, $\partial f_i/\partial \lambda_j$ exist, are continuous in D_λ, and satisfy there the Cauchy-Riemann equations—which are a consequence of the equations

$$\begin{aligned}\frac{\partial f}{\partial w} &= \frac{\partial f_1}{\partial w_1}+i\frac{\partial f_2}{\partial w_1}=\frac{\partial f_2}{\partial w_2}-i\frac{\partial f_1}{\partial w_2},\\ \frac{\partial f}{\partial \lambda} &= \frac{\partial f_1}{\partial \lambda_1}+i\frac{\partial f_2}{\partial \lambda_1}=\frac{\partial f_2}{\partial \lambda_2}-i\frac{\partial f_1}{\partial \lambda_2}.\end{aligned} \tag{4}$$

Now, writing $w=w_1+iw_2$, $\lambda=\lambda_1+i\lambda_2$, *etc.*, we replace the single complex equation (1) by the equivalent system (2). By hypothesis this system has, for $\bar{\lambda}=\bar{\lambda}_1+i\bar{\lambda}_2$, a solution $(w_1(t), w_2(t))$—where $w_1(t)+iw_2(t)=w(t)$—defined on I. Moreover, by our preceding remarks the remaining hypotheses of Theorem II are also satisfied. Hence there exists $\delta_1>0$ such that for any point $(t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)\in U_{\delta_1}$, there exists a unique solution $W_i(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)$ of (2) on $a\leqq t\leqq b$ satisfying

$$W_i(t^0; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0) = w_i^0 \qquad (i=1, 2).$$

If we now set

$$W(t; t^0, w^0, \lambda^0) = W_1(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0) + iW_2(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)$$

and $\delta=\delta_1/\sqrt{2}$, then the existence, uniqueness and continuity properties of Theorem I follow immediately from the above on noting that

$$|w_1^0 - w_1(t^0)| + |w_2^0 - w_2(t^0)| + |\lambda_1^0 - \bar{\lambda}_1| + |\lambda_2^0 - \bar{\lambda}_2|$$
$$\leqq \sqrt{2}(|w^0 - w(t^0)| + |\lambda^0 - \bar{\lambda}|).$$

It remains, however, to prove that—for each fixed $(t, t^0) - W(t; t^0, w^0, \lambda^0)$ is an analytic function of the two complex variables w^0, λ^0 for $(w^0, \lambda^0) \in U_b$. To prove this, we note that

$$(5) \quad W_i(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0) = w_i^0 + \int_{t^0}^{t} f_i(t, W_1, W_2, \lambda_1^0, \lambda_2^0)dt \qquad (i = 1, 2),$$

for $(t, t^0, w^0, \lambda^0) \in V_b$. Hence, differentiating, we have

$$(6) \quad \begin{aligned} \frac{\partial W_1}{\partial w_1^0} &= 1 + \int_{t^0}^{t} \left\{ \frac{\partial f_1}{\partial w_1} \frac{\partial W_1}{\partial w_1^0} + \frac{\partial f_1}{\partial w_2} \frac{\partial W_2}{\partial w_1^0} \right\} dt, \\ \frac{\partial W_1}{\partial w_2^0} &= \int_{t^0}^{t} \left\{ \frac{\partial f_1}{\partial w_1} \frac{\partial W_1}{\partial w_2} + \frac{\partial f_1}{\partial w_2} \frac{\partial W_2}{\partial w_2^0} \right\} dt, \\ \frac{\partial W_2}{\partial w_1^0} &= \int_{t^0}^{t} \left\{ \frac{\partial f_2}{\partial w_1} \frac{\partial W_1}{\partial w_1^0} + \frac{\partial f_2}{\partial w_2} \frac{\partial W_2}{\partial w_1^0} \right\} dt, \\ \frac{\partial W_2}{\partial w_2^0} &= 1 + \int_{t^0}^{t} \left\{ \frac{\partial f_2}{\partial w_1} \frac{\partial W_1}{\partial w_2^0} + \frac{\partial f_2}{\partial w_2} \frac{\partial W_2}{\partial w_2^0} \right\} dt. \end{aligned}$$

It follows that $X(t) \equiv (\partial/\partial w_1^0) W_1(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)$, $Y(t) \equiv \partial W_2/\partial w_1^0$, $Z(t) \equiv \partial W_1/\partial w_2^0$, $V(t) \equiv \partial W_2/\partial w_2^0$ are solutions of the linear system

$$(7) \quad \begin{aligned} X' &= \frac{\partial f_1}{\partial w_1} X + \frac{\partial f_1}{\partial w_2} Y + 0 \cdot Z + 0 \cdot V, \quad Z' = 0 \cdot X + 0 \cdot Y + \frac{\partial f_1}{\partial w_1} Z + \frac{\partial f_1}{\partial w_2} V, \\ Y' &= \frac{\partial f_2}{\partial w_1} X + \frac{\partial f_2}{\partial w_2} Y + 0 \cdot Z + 0 \cdot V, \quad V' = 0 \cdot X + 0 \cdot Y + \frac{\partial f_2}{\partial w_1} Z + \frac{\partial f_2}{\partial w_2} V, \end{aligned}$$

on the interval $a \leqq t \leqq b$. Here,

$$\frac{\partial f_1}{\partial w_1} = \frac{\partial}{\partial w_1} f_1(t, W_1(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0), W_2(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0), \lambda_1^0, \lambda_2^0),$$

etc., and we are holding $t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0$ fixed. This solution system satisfies the initial conditions $X(t^0) = 1$, $Y(t^0) = 0$, $Z(t^0) = 0$, $V(t^0) = 1$, by (6).

We now assert that the system (7) also has a solution system

$$x(t) = \frac{\partial}{\partial w_2^0} W_2(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0), \quad y(t) = -\partial W_1/\partial w_2^0,$$
$$z(t) = -\partial W_2/\partial w_1^0, \quad v(t) = \partial W_1/\partial w_1^0.$$

This follows from the fact that $\partial f_1/\partial w_1 \equiv \partial f_2/\partial w_2$, $\partial f_1/\partial w_2 \equiv -\partial f_2/\partial w_1$, according

to the first of equations (4). Thus, for example, the first of equations (7) may be rewritten as $X' = (\partial f_1/\partial w_2)X - (\partial f_2/\partial w_1)Y$. Since

$$x'(t) = \frac{d}{dt}\left(\frac{\partial W_2}{\partial w_2^0}\right) = \frac{\partial f_2}{\partial w_1}\frac{\partial W_1}{\partial w_2^0} + \frac{\partial f_2}{\partial w_2}\frac{\partial W_2}{\partial w_2^0} = -\frac{\partial f_2}{\partial w_1}y(t) + \frac{\partial f_2}{\partial w_2}x(t),$$

we see that our new system satisfies the first of equations (7). Similarly one may verify that all of equations (7) are satisfied. Moreover, we have $x(t^0) = 1$, $y(t^0) = 0$, $z(t^0) = 0$, $v(t^0) = 1$. Hence, by the uniqueness theorem for systems of linear differential equations ([1], Theorem 5.1) we conclude that $X \equiv x$, $Y \equiv y$, $Z \equiv z$, $V \equiv v$ on $a \leqq t \leqq b$. That is, for arbitrary $(t, t^0, w^0, \lambda^0) \in V_3$, we have

$$\frac{\partial W_1}{\partial w_1^0} = \frac{\partial W_2}{\partial w_2^0}, \qquad \frac{\partial W_2}{\partial w_1^0} = -\frac{\partial W_1}{\partial w_2^0}. \tag{8}$$

Similarly, by differentiating equations (5) with respect to λ_1^0 and λ_2^0, we may conclude that

$$\frac{\partial W_1}{\partial \lambda_1^0} \equiv \frac{\partial W_2}{\partial \lambda_2^0}, \qquad \frac{\partial W_2}{\partial \lambda_1^0} \equiv -\frac{\partial W_1}{\partial \lambda_2^0}. \tag{9}$$

Since all the derivatives appearing in (8) and (9) are continuous and since W is itself continuous on V_3, it follows that for each fixed (t, t^0), W is an analytic function of (w^0, λ^0) for $(w^0, \lambda^0) \in U_3$.

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1. E. Coddington and N. Levinson, Theory of Ordinary Differential Equations, New York, 1955.
2. É. Picard, Traité d'Analyse, Tome II, 3rd ed., Paris, 1926.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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A SIMPLE ILLUSTRATION OF OPERATIONAL METHODS

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Let A and X_0 be (complex) constant matrices of orders n-by-n and 1-by-n, respectively, and consider the nth order linear homogeneous system

$$X'(t) = X(t)A, \qquad X(0) = X_0, \tag{1}$$

of differential equations with constant coefficients. It is well-known that the

solution of this system can be given by

$$X(t) = X_0[F_0(t)I + F_1(t)A + \cdots + F_{n-1}(t)A^{n-1}], \tag{2}$$

where each scalar function F_k satisfies

$$c_0F_k(t) + c_1F_k'(t) + \cdots + c_{n-1}F_k^{(n-1)}(t) + F_k^{(n)}(t) = 0, \qquad F_k^{(i)}(0) = \delta_k^i, \tag{3}$$

$i=0, \cdots, n-1$, where $f(\lambda)=c_0+c_1\lambda+\cdots+c_{n-1}\lambda^{n-1}+\lambda^n$ is the characteristic polynomial of A and δ_k^i is the Kronecker delta. The following demonstration of this result may prove of interest in the classroom to students being first introduced to matrix algebra and Laplace transform methods.

The Laplace transform of (1) can be represented as

$$x(s)(sI - A) = X_0. \tag{4}$$

Assume that there are scalar functions $f_0, f_1, \cdots, f_{n-1}$ such that

$$(sI - A)[f_0(s)I + f_1(s)A + \cdots + f_{n-1}(s)A^{n-1}] = I. \tag{5}$$

Since $f(A)=c_0I+c_1A+\cdots+c_{n-1}A^{n-1}+A^n=0$, equation (5) will be satisfied if

$$sf_k(s) - f_{k-1}(s) + c_kf_{n-1}(s) = 0 \tag{6}$$

for $k=0, \cdots, n-1$, where $f_{-1}(s)=1$. If (6) is multiplied by s^k and the resulting n equations are added together, it follows that $f(s)\cdot f_{n-1}(s)=1$. Hence, using equation (6) again,

$$f_k(s) = \frac{1}{f(s)} \sum_{j=k}^{n-1} c_{j+1}s^{j-k}, \qquad k = 0, \cdots, n-1, \tag{7}$$

where $c_n=1$. By combining the preceding results, it follows that $x(s)=X_0[f_0(s)I+f_1(s)A+\cdots+f_{n-1}(s)A^{n-1}]$. This in turn gives equation (2) where F_k is the inverse Laplace transform of the rational function f_k.

Furthermore, it is to be observed that equation (7) can also be written as

$$f(s)\cdot f_k(s) - \sum_{i=0}^{n-1}\left(\sum_{j=i}^{n-1} c_{j+1}s^{j-i}\right)\delta_k^i = 0.$$

By interchanging the order of summation and by using the polynomial expansion of f, this becomes

$$c_0\cdot f_k(s) + \sum_{j=0}^{n-1} c_{j+1}\left[s^{j+1}f_k(s) - \sum_{i=0}^{j} s^{j-i}\delta_k^i\right] = 0.$$

Finally, by recalling the algebraic expression giving the Laplace transform of $F_k^{(i)}$ in terms of f_k, it is clear that if F_k satisfies system (3), then its Laplace transform is given by equation (7).

AN APPLICATION OF ROTATION FORMULAS

R. LARIVIERE AND G. LOUIS, University of Illinois, Chicago

The normal form of the equation of a straight line is taught students of plane analytical geometry chiefly to enable them to write the distance from an oblique line to a point with little computation. The topic has long been a stumbling block, especially when the direction cosines of the lines are used in plane analytic geometry in preparation for their use in solid [1]. Many authors have attempted to substitute other methods of deriving the line-to-point distance [2, 3, 4]. Often these have met with indifferent success except in the hands of their originators. In at least one case the struggle has been abandoned, and the authors have returned to a presentation of the normal form as an intermediate step in the derivation of the line-to-point formula [5].

The following method of deriving this formula is an interesting application of rotation of axes. A rotation of the axes through an angle θ equal to the smallest counter-clockwise angle between the positive y-axis and the undirected line transforms the equation of the line $Ax+By+C=0$ to $x=p$. Substituting the rotation formulas in the equation and eliminating the term in y we find that $p = -C/(\pm\sqrt{A^2+B^2})$, $\tan\theta = B/A$, $\sin\theta = B/(\pm\sqrt{A^2+B^2})$, and $\cos\theta = A/(\pm\sqrt{A^2+B^2})$. The distance from the new y-axis to any point $P(x_1, y_1)$ is the new abscissa of P, $x_1\cos\theta+y_1\sin\theta$. And the distance from the line $x=p$ to the point P is $x_1\cos\theta+y_1\sin\theta-p$, namely $(Ax_1+By_1+C)/(\pm\sqrt{A^2+B^2})$.

With θ defined as above, the convention of signs implied is that the sign of the perpendicular from the origin to the line agrees with that of the y-intercept. Since $\theta<180°$ we are able to choose the sign of the radical as in the traditional procedures which use this convention.

If the convention preferred is that the perpendicular distance from the origin to the line should be positive in all four quadrants, the angle of rotation is $(180°+\theta)$ when the y-intercept is negative, or when the y-intercept is zero and the x-intercept is negative. As before $\tan\theta=B/A$, and $p=-C/(\pm\sqrt{A^2+B^2})$, but $p\geq 0$. The distance from the line to the point P is $(Ax_1+By_1+C)/(\pm\sqrt{A^2+B^2})$.

The sign before the radical is now chosen as in the traditional procedures which use the convention that the sign of the perpendicular from the origin to the line is always positive. We may adopt these procedures since, when $C=0$ and the line passes through the origin, θ may be taken within the range $180°>\theta>0$.

References

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2. J. W. Cell, Analytic Geometry, New York, 1951, p. 42.
3. N. McCoy and R. Johnson, Analytic Geometry, New York, 1955, p. 52.
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COMPUTATION OF COMMON LOGARITHMS BY REPEATED SQUARINGS

O. E. BROWN, The Babcock and Wilcox Company

1. Introduction. A method for the computation of common logarithms has been derived by the writer which is so elementary that it can be understood by high school students and has the additional merit of lending itself to easy programming for modern digital computers. The method, like that described by Shanks [1] and Ostrowski [2] utilizes the arithmetic properties of the individual logarithm rather than the analytic properties of the logarithmic function.

2. Derivation. The entire method rests squarely upon the following:

THEOREM. *If* $N_1, N_2, \cdots$ *and* $a_1, a_2, \cdots$ *be two sequences of nonnegative real numbers such that* $1 \leqq N_i < 10$ *and* $a_i = 0$ *or* $a_i = 1$, *for every* i;

$$\text{(1)} \qquad \text{if } N_i^2 < 10, \qquad N_{i+1} = N_i^2, \qquad a_i = 0$$

and

$$\text{(2)} \qquad \text{if } N_i^2 \geqq 10, \qquad N_{i+1} = N_i^2/10, \qquad a_i = 1;$$

then

$$\text{(3)} \qquad \log_{10} N_1 = a_1/2 + a_2/2^2 + \cdots.$$

Proof. Let the binary representation of $\log_{10} N_1$ consist of the bits $b_1, b_2, \cdots$, *i.e.*,

$$\log_{10} N_1 = b_1/2 + b_2/2^2 + \cdots,$$

where $b_i = 0$ or 1. Then*

$$\text{(4)} \qquad N_1 = \exp_{10} (b_1/2 + b_2/2^2 + \cdots)$$

and squaring gives

$$\text{(5)} \qquad N_1^2 = \exp_{10} (b_1 + b_2/2 + \cdots).$$

Clearly, if $N_1^2 \geqq 10$ then $b_1 = 1$, while if $N_1^2 < 10$ then $b_1 = 0$ so that $b_1 = a_1$, and upon division of both sides of (5) by $\exp_{10} b_1$, we have

$$\text{(6)} \qquad \frac{N_1^2}{\exp_{10} b_1} = N_2 = \exp_{10} (b_2/2 + b_3/2^2 + \cdots).$$

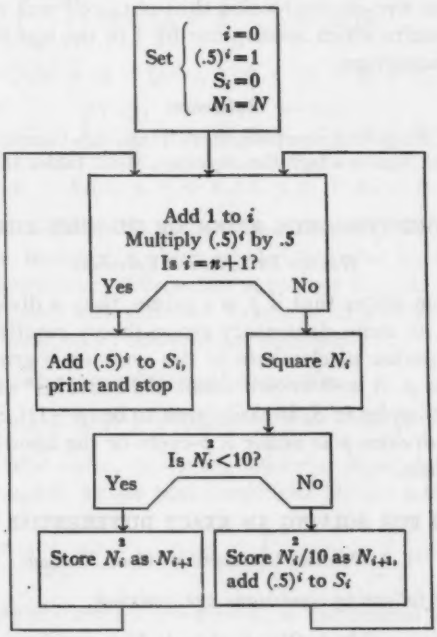
Since by (6) we have N_2 expressed in terms of $b_2, b_3, \cdots$, in the same way as (5) expresses N_1 in terms of $b_1, b_2, \cdots$, the squaring of N_2 will show that

* We write $\exp_{10} x$ for 10^x.

$b_2 = a_2$. Continuation establishes the identity of the sequence b_i with the sequence a_i and the theorem is proved.

3. Program. In order to write a computer program for the method of this paper it is only necessary to calculate each successive N_i by squaring its predecessor and dividing by 10 if need be. The a_i are selected according to (1) and (2) and $\log_{10} N_1$ accumulated by (3). A suggested block diagram is shown in Table 1.

TABLE 1



4. Accuracy. There are plainly two sources of error in the calculation. One of these is the round-off error in the squaring process. We are making what amounts to a floating decimal calculation for which estimation of errors is difficult, to say the least. However, a few trials seems to indicate that if k figures are carried in the N_i then the $\log_{10} N_1$ obtained will be correct to k figures. To be quite safe we may carry $2k$ figures of each N_i in calculating $\log_{10} N_1$ to k figures.

The other type of error which arises is that due to the necessary truncation of the series (3). Suppose that this truncation takes place after n terms, so that the deleted terms are

$$a_{n+1}/2^{n+1} + a_{n+2}/2^{n+2} + \cdots. \tag{7}$$

Since each of the a's is either 0 or 1, we may treat them as random variables whose two values have equal probabilities. The expected value of each a_i then becomes $1/2$ and the truncation error (7) has the expected value $1/2^{n+1}$.

The writer has programmed the Electrodata E101 to employ this method with $n=20$. Peculiarities of that machine dictate that 10 figures are kept of each N_i. To the truncated series we have added the expected remainder term $1/2^{21}$ or .000 000 4768 as well as the round-off correction .000 005 and have printed out the resulting five-decimal logarithm. We have not, as yet, observed a case in which the five-decimal value thus obtained was wrong. We have encountered a few results which are in error by 1 in the last figure when we tried stopping after 18 squarings.

References

1. Daniel Shanks, A logarithm algorithm, Math. Tables Aids Comput., vol. 8, 1954, pp. 60–64.
2. A. M. Ostrowski, Note on a logarithm algorithm, Math. Tables Aids Comput., vol. 9, 1955, pp. 65–68.

A GROUP-THEORETIC PROOF OF WILSON'S THEOREM

WALTER FEIT, Cornell University

Wilson's theorem states that if p is a prime, then p divides $(p-1)!+1$. We give a proof based on some elementary group theory results.

Let n be the number of elements in the symmetric group S_p on p letters, whose order divides p. A well-known result of Frobenius* states that p divides n. The number of p-cycles in S_p is easily seen to be $(p-1)!$. Since every element in S_p whose order divides p is either a p-cycle or the identity, $n=(p-1)!+1$. The proof is complete.

A METHOD FOR SOLVING AN EXACT DIFFERENTIAL EQUATION

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Suppose that the following conditions are satisfied:

$$P(x, y, z)dz + Q(x, y, z)dy + R(x, y, z)dz = 0,$$

(1)
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y};$$

(2) $Q = Q_1 + Q_2$ *and* $\dfrac{\partial Q_2}{\partial x} = 0$, $\quad R = R_1 + R_2$ *and* $\dfrac{\partial R_2}{\partial x} = 0$, $\quad \dfrac{\partial R_2}{\partial y} = 0$;

(3) $$Q_1(x_0, y, z) = 0 \text{ and } \int_{y_0}^{y} \frac{\partial Q_2}{\partial z}\, dy = R_1(x_0, y, z).$$

Then the solution of equation (1) *is given by*

* See W. Burnside, Theory of Groups of Finite Order, Cambridge University, 1911, p. 49.

$$F(x, y, z) = C \qquad (C = \text{constant}), \tag{4}$$

where

$$F = \int_{x_0}^{x} P\,dx + \int_{y_0}^{y} Q_2\,dy + \int_{z_0}^{z} R_3\,dz. \tag{5}$$

Proof. We shall show that $(\partial F/\partial x) = P$, $(\partial F/\partial y) = Q$ and $(\partial F/\partial z) = R$. Observe from equation (5) that $(\partial F/\partial x) = P$. Also

$$\frac{\partial F}{\partial y} = \int_{z_0}^{z} \frac{\partial P}{\partial y}\,dx + Q_2 = \int_{x_0}^{x} \frac{\partial Q_1}{\partial x}\,dx + Q_2$$

$$= Q_1(x, y, z) - Q_1(x_0, y, z) + Q_2 = Q_1 + Q_2 = Q.$$

$$\frac{\partial F}{\partial z} = \int_{z_0}^{z} \frac{\partial P}{\partial z}\,dz + \int_{y_0}^{y} \frac{\partial Q_2}{\partial z}\,dy + R_3 = \int_{x_0}^{x} \frac{\partial R_1}{\partial x}\,dx + R_1(x_0, y, z) + R_2$$

$$= R_1(x, y, z) - R_1(x_0, y, z) + R_1(x_0, y, z) + R_2 = R_1 + R_2 = R.$$

Remark. Almost all problems given in textbooks satisfy conditions (3) with $x_0 = y_0 = 0$. At times, however, a given exact differential equation may be separated into two or more exact differential equations in each of which the above method may be applied with different values of x_0 and y_0. The sum of these solutions gives the solution of our original equation.

Example 1. Solve the exact differential equation

$$(z^2y^2 - z^2)x\,dx + (x^2z^2 + z^2)y\,dy + (x^2y^2 - x^2 + y^2 - 1)z\,dz = 0.$$

Observe that $P = (z^2y^2 - z^2)x$, $Q_1 = x^2z^2y$, $Q_2 = z^2y$, $R_1 = (x^2y^2 - x^2 + y^2)z$ and $R_2 = -z$. Taking $x_0 = y_0 = 0$, we see that conditions (3) are satisfied. Hence

$$F = \int_0^x (z^2y^2 - z^2)x\,dx + \int_0^y z^2y\,dy - \int_0^z z\,dz.$$

Therefore, $(1/2)(x^2y^2z^2 - x^2z^2 + z^2y^2 - z^2) = C$, is the desired solution.

Example 2. Solve the exact differential equation

$$(e^z \cos y + yz)dx + (xz - e^z \sin y)dy + (xy + z)dz = 0.$$

This equation may be separated into two exact equations

$$e^z \cos y\,dx - e^z \sin y\,dy = 0, \tag{6}$$

$$yz\,dx + xz\,dy + (xy + z)dz = 0. \tag{7}$$

For equation (6), we have $P = e^z \cos y$, $Q_1 = e^z \sin y$, $Q_2 = R_1 = R_2 = 0$. Conditions (3) are satisfied for $x_0 = -\infty$ and $y_0 = 0$. Hence $F_1 = \int_{-\infty}^{x} e^z \sin y\,dx = e^z \sin y$. For equation (7), $P = yz$, $Q_1 = xz$, $Q_2 = 0$, $R_1 = xy$, $R_2 = z$. Conditions (3) are satisfied for $x_0 = y_0 = 0$. Thus $F_2 = \int_0^x yz\,dx + \int_0^z z\,dz = xyz + \frac{1}{2}z^2$. Hence the solution is $e^z \cos y + xyz + \frac{1}{2}z^2 = C$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1301. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the angles of a triangle ABC are consecutive terms of a geometric progression of common ratio 3, then

$$\cos B \cos C + \cos C \cos A + \cos A \cos B = -1/4.$$

E 1302. *Proposed by M. S. Klamkin, AVCO, Research and Development, Lawrence, Mass.*

A square is divided into two parts by an arbitrary diameter through its center. Determine the locus of the centroid of one of the equal areas.

E 1303. *Proposed by M. J. Hellman, Rutgers University*

Solve the differential equation $dy/dx = \Re(z^n)/\Im(z^n)$, where $z = x+iy$, n is a positive integer, $\Re(z^n)$ denotes the real part of z^n, and $\Im(z^n)$ denotes the imaginary part of z^n.

E 1304. *Proposed by W. B. Carver, Cornell University*

Let A_1, A_2, A_3 be the vertices of any triangle and let the arc of the circumcircle from A_i to A_j be trisected by the points T_{ij} and T_{ji}, T_{ij} adjacent to A_i and T_{ji} adjacent to A_j, $i, j = 1, 2, 3$. Further, let A_1T_{32} and A_3T_{12} intersect in P_2, A_1T_{23} and A_2T_{13} intersect in P_3. Show that $T_{31}T_{21}$ is parallel to P_2P_3.

E 1305. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Given a positive integer n and a symmetric 3×3 matrix $A = (a_{ij})$ with integral elements, when does there exist an $n\times3$ matrix B such that $B^*B = A$ (here B^* denotes the transpose of B) and every element of B is either 0 or 1? Describe a method for finding all solutions B.

SOLUTIONS

A Line Touching a Sphere and a Cube

E 1271 [1957, 432]. *Proposed by Ward Cheney and A. A. Goldstein, Convair-Astronautics, San Diego, California*

In E_3 let L denote a line not through the origin and not parallel to any coordinate plane. Does the point of contact of L with the smallest sphere having

center at the origin necessarily lie in the same octant as the point of contact of L with the smallest cube having center at the origin and edges parallel to the coordinate axes?

Solution by W. B. Carver, Cornell University. No. For example, let the line L be $(x-1)/4=-(y-3)/4=z-8$. The sphere touching this line is the sphere $x^2+y^2+z^2=74$, the point of contact being (1, 3, 8). The cube touching this line is the cube bounded by the planes $x=\pm 7$, $y=\pm 7$, $z=\pm 7$, the point of contact being $(-3, 7, 7)$.

Also solved by Leon Bankoff, Michael Goldberg, A. R. Hyde, C. S. Ogilvy, and the proposers. The proposers pointed out that the answer to the corresponding problem in E_2 is "yes," and is also "yes" in E_3 if L denotes a plane, or more generally in E_n if L denotes a hyperplane.

An Inequality for the Triangle

E 1272 [1957, 432]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If A, B, C are the angles of a triangle, show that

$$(\sin A/2 + \sin B/2 + \sin C/2)^2 \leqq \cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2.$$

Solution by Michael Goldberg, Washington, D. C. Since $\sin\theta$ increases with θ for $0<\theta<\pi/2$, but the rate of increase decreases with θ, the maximum value of $\sin A/2+\sin B/2+\sin C/2$ for $A/2+B/2+C/2\leqq\pi/2$ occurs when $A/2=B/2=C/2$. Similarly, since $\cos^2\theta$ decreases with θ for $0<\theta<\pi/2$, but the rate of decrease increases with θ, the minimum value for $\cos^2 A/2+\cos^2 B/2+\cos^2 C/2$ for $A/2+B/2+C/2\leqq\pi/2$ is attained when $A/2=B/2=C/2$. But for $A/2=B/2=C/2=\pi/6$, the right and left members are equal. Hence the sought relation.

Also solved by Leon Bankoff, A. P. Boblétt, D. A. Breault, Calvin Foreman, A. R. Hyde, P. C. Keller, M. A. Laframboise, D. C. B. Marsh, and the proposer.

A Criterion for a Polygon to be Regular

E 1273 [1957, 432]. *Proposed by Alan Wayne, Baldwin, New York*

Prove or disprove the following proposition (which is true for $n=3$ and $n=4$): A plane n-gon with incircle of radius r and circumcircle of radius R is regular if and only if $r=R\cos(\pi/n)$.

Solution by Howard Eves, University of Maine. Let p be the perimeter of the n-gon. Now it is known that of all n-gons inscribed in a circle of radius R, the regular n-gon has the maximum perimeter. Hence $p\leqq 2nR\sin(\pi/n)$, equality holding if and only if the n-gon is regular. It is also known that of all n-gons circumscribed about a circle of radius r, the regular n-gon has the minimum perimeter. Hence $p\geqq 2nr\tan(\pi/n)$, equality holding if and only if the n-gon is regular. It now follows that $r/R\leqq\cos(\pi/n)$, equality holding if and only if the n-gon is regular.

The proposition is somewhat similarly established in Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Springer, 1953) pp. 6–8.

Also solved by Michael Goldberg (paraphrasing Tóth's proof), and D. C. B. Marsh. William Moser also gave the Tóth reference. Late solution by D. A. Breault.

An Inequality from Information Theory

E 1274 [1957, 432]. *Proposed by P. G. Kirmser, Kansas State College*

Given $p_i>0$, $q_i>0$, and $\sum_{i=1}^{n} p_i=\sum_{i=1}^{n} q_i$, show that

$$\sum_{i=1}^{n} p_i \ln p_i \geqq \sum_{i=1}^{n} p_i \ln q_i.$$

Solution by Bernhard Marzetta, Basle, Switzerland. For $t>0$ we have

$$t \ln t = \int_1^t (t/s)ds \geqq \int_1^t ds = t - 1.$$

Multiplying both sides of $(p_i/q_i) \ln (p_i/q_i) \geqq p_i/q_i - 1$ by q_i, and summing over i, we get $\sum_{i=1}^{n} p_i \ln (p_i/q_i) \geqq 0$. Equality can occur only when $p_i=q_i$ for all i.

Also solved by J. L. Botsford, L. R. Bragg, P. L. Duren, Calvin Foreman, Michael Goldberg, D. S. Greenstein, A. R. Hyde, M. S. Klamkin, Morton Kupperman, L. M. Lewandowski, Leon Lifton, D. C. B. Marsh, S. C. Port, D. A. Robinson, H. W. E. Schwerdtfeger, D. L. Smith, Chih-yi Wang, and the proposer. Late solution by R. H. Hou.

Kupperman remarked that this result is well known in information theory, where the p_i's and q_i's are probabilities, so that $\sum_{i=1}^{n} p_i=\sum_{i=1}^{n} q_i=1$. He pointed out that a proof may be found in L. Brillouin, *Science and Information Theory* (Academic Press, 1956), pp. 13–14.

Beta Functions

E 1275 [1957, 504]. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development, Lawrence, Massachusetts*

Solve for x:

$$\int_0^x s^{2/3}(1-s)^{4/3}ds = \int_0^1 t^{2/3}(1+t)^{-6}dt.$$

Solution by Calvin Foreman, Baker University. Set $t=s/(1-s)$ to obtain

$$\int_0^x s^{2/3}(1-s)^{4/3}ds = \int_0^{1/2} s^{2/3}(1-s)^{4/3}ds.$$

Since the integral on the left is a monotonically increasing function of x, the only solution is $x=1/2$.

Also solved by Julian Braun, Tien Chi Chen, Thomas Erber, S. H. Greene, Nathaniel Grossman, Emil Grosswald, J. R. Hendricks, J. R. Holdsworth, A. R. Hyde, E. S. Keeping, A. G. Konheim, D. C. B. Marsh, Bernard Marzetta, D. L. Muench, W. Renga, E. T. Sheffield, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by W. S. Lawton.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4773. *Proposed by Paul Erdös, University of Toronto*

Let $n_1 \leqq n_2 \leqq \cdots$ be a sequence of positive integers. Assume $n_k^{1/2^k} \to \infty$. Show that $\sum_{k=1}^{\infty} 1/n_k$ is irrational. (It is easy to construct $\sum 1/n_k$ rational with $n_k > A^{2^k}$ for every fixed A.)

4774. *Proposed by Donald Greenspan, University of Maryland*

Give an example of a simple closed curve in Euclidean 3-space which satisfies the following two conditions: 1. No three points of the curve lie on a straight line. 2. No four points of the curve lie on a circle.

4775. *Proposed by Leonard Carlitz, Duke University*

Prove the formulas:

$$(1) \qquad \sum_{r=0}^{m} \frac{A_r A_{m-r} A_{n-r}}{A_{m+n-r}} \frac{2m+2n-4r+1}{2m+2n-2r+1} = 1,$$

$$(2) \qquad \sum_{r=0}^{m} (-2)^r \frac{A_r A_{m-r} A_{n-r} A_{k-r}}{A_{m+n-r}} \frac{4k-4r+1}{4k-2r+1} = A_{m/2} A_{n/2},$$

where $m \leqq n$, $A_r = 1 \cdot 3 \cdots (2r-1)/r!$, $A_0 = 1$, $A_x = 0$ for x not an integer, and, in (2), $m+n=2k$.

4776. *Proposed by D. J. Newman, AVCO Research & Development*

If $|\alpha| < 1/e$, then $\sum_{n=0}^{\infty} (z+\alpha n)^n/n!$ represents an entire function.

4777. *Proposed by R. P. Pakshirajan, Indian Statistical Institute, Calcutta*

Prove that

$$\lim_{n\to\infty} \frac{n}{2^{2n}} \left\{ \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 \right\} = \frac{2}{\pi\sqrt{3}}.$$

SOLUTIONS

Divergent Integrals

4728 [1957, 201]. *Proposed by R. P. Boas, Jr., Northwestern University*

A. M. Rodov has propounded a proof that if $f(x)$ is continuous and the first of the following integrals converges, then the second diverges

$$\int_1^\infty f(x)dx, \qquad \int_1^\infty x^{-2}\{f(x)\}^{-1}dx.$$

(a) Construct a counter-example. (b) More generally, show that if $g(x)$ and $\phi(x)$ are positive and $\int_1^\infty \phi(x)dx$ diverges, then at least one of

$$\int_1^\infty \phi(x)g(x)dx \quad \text{and} \quad \int_1^\infty \{\phi(x)/g(x)\}dx$$

diverges.

I. *Solution by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.* (a) With $f(x)=(-1)^n/x$, $n=[\sqrt{x}]$, both integrals are easily seen to be convergent.

(b) By the Schwartz inequality,

$$\int_1^\infty \phi(x)g(x)dx \cdot \int_1^\infty \phi(x)/g(x)dx \geqq \left[\int_1^\infty \phi(x)dx\right]^2.$$

When $\int_1^\infty \phi(x)dx$ diverges, then at least one of the two integrals on the left must diverge.

II. *Solution by D. P. Squier and David Zeitlin, Remington Rand Univac, St. Paul, Minn.* (a) The function $f(x)=\sin x/x|\sin x|^{1/2}$ gives a counter-example.

(b) Since $g(x)+1/g(x)\geqq 2$, we have

$$\int_1^m \phi(x)g(x)dx + \int_1^m \frac{\phi(x)}{g(x)}dx = \int_1^m \phi(x)\left[g(x)+\frac{1}{g(x)}\right]dx \geqq 2\int_1^m \phi(x)dx.$$

Letting $m\to\infty$, we conclude that at least one of the given integrals diverges.

Also solved by R. P. Agnew, Robert Breusch, M. F. Friedell, Ben Fusaro, R. R. Goldberg, A. R. Harvey, G. G. Lorentz, Alexander Peyerimhoff, Chih-yi Wang, and the proposer.

Irrational Numbers

4729 [1957, 201]. *Proposed by Paul Erdös, University of Toronto*

Let b_k, $k=1, 2, \cdots$, be any sequence of nonnegative integers such that $\limsup b_k^{1/k}<2$. Assume further that $\sum_{k=1}^n b_k\to\infty$, $\liminf \sum_{k=1}^n b_k/n=0$. Prove that $\sum_{k=1}^\infty b_k/2^k$ is irrational.

Solution by Robert Breusch, Amherst College. Let b_k $(k=1, 2, \cdots)$ be a sequence of nonnegative integers such that

(1) $\limsup b_k^{1/k} < 2$, (2) $\sum_{k=1}^{n} b_k \to \infty$,

(3) $\sum_{k=1}^{\infty} b_k/2^k = A/B$, A and B positive integers.

It will be shown that

(4) $$\liminf \frac{1}{n} \sum_{k=1}^{n} b_k > 0.$$

We may assume $B=1$, for the sequence (Bb_k) will also satisfy (1) and (2), and $\liminf \sum_{k=1}^{n} (Bb_k)/n > 0$ implies (4).

(1) assures the existence of two positive numbers r and k_0 such that

(5) $$r < 1, \quad \text{and} \quad b_k < (2r)^k \quad \text{for} \quad k \geqq k_0.$$

Let s be an integer >1 such that

(6) $$2r^s < 1, \quad \text{and} \quad 2r^s < \left(\frac{1-r}{2r}\right)^{1/k_0}.$$

Let n be a positive integer $\geqq sk_0$, and

(7) $$m = [n/s]; \quad \text{thus} \quad m \geqq k_0.$$

For every positive integer t, let $C_t = \sum_{k=t+1}^{\infty} b_k/2^{k-t}$; then by (3) C_t is an integer, and by (2) $C_t > 0$; thus

(8) $$C_t \geqq 1.$$

Let $t \leqq m$. Then certainly $t < n$, and

(9) $$C_t = \sum_{k=t+1}^{\infty} b_k/2^{k-t} = \sum_{k=t+1}^{n} b_k/2^{k-t} + \sum_{k=n+1}^{\infty} b_k/2^{k-t}.$$

By (5), (6) and (7),

$$\sum_{k=n+1}^{\infty} b_k/2^{k-t} < 2^t \sum_{k=n+1}^{\infty} r^k \leqq 2^m \cdot r^n \cdot \frac{r}{1-r} \leqq (2r^s)^m \cdot \frac{r}{1-r} < \frac{1}{2}.$$

Therefore, by (8) and (9),

$$\sum_{k=t+1}^{n} b_k/2^{k-t} > \tfrac{1}{2}$$

for every positive integer $t \leqq m$.

Adding these inequalities for $t = 1, 2, \cdots, m$, we obtain

$$\frac{1}{2} m < \sum_{t=1}^{m} \sum_{k=t+1}^{n} b_k/2^{k-t} < \sum_{k=2}^{n} b_k \sum_{t=1}^{k-1} 1/2^{k-t} < \sum_{k=1}^{n} b_k.$$

Therefore $\sum_{k=1}^{n} b_k/n > m/2n = [n/s]/2n$, and thus

$$\liminf \frac{1}{n} \sum_{k=1}^{n} b_k \geq \frac{1}{2s}.$$

Also solved by G. G. Lorentz.

Probabilities in an Imperfect Sorting Process

4731 [1957, 201]. *Proposed by D. S. Stoller, Los Angeles, Calif.*

Consider an imperfect sorting process acting on a very large number of items, each of which belongs in one and only one of k bins. Let Q_i represent the probability that if an item belongs in bin i, it is sorted into bin i, and if it does not belong in bin i, it is sorted into some other unspecified bin. Find Q, the probability that an item is sorted into the correct bin.

Solution by A. R. Hyde, West Hartford, Conn. If P_i and R_i denote, respectively, the probabilities that an item belongs in bin i and that it is assigned to bin i, then

$$Q_i = P_iR_i + (1 - P_i)(1 - R_i).$$

The summation over k is $\sum Q_i = 2\sum P_iR_i + k - 2$. Hence

$$Q \equiv \sum P_iR_i = 1 - \tfrac{1}{2}(k - \sum Q_i).$$

Also solved by D. S. Greenstein and the proposer.

Concurrent Lines

4735 [1957, 277]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i, and the common point of the lines A_iB_{i+1}, B_iA_{i+1} by C_{i+3}. Then the five lines A_iC_i have a point D in common.

Solution by E. J. F. Primrose, The University, Leicester, England. There is a unique polarity P for which each A_i is the pole of the opposite side a_i (Coxeter, *The Real Projective Plane*, 5.65). We consider the 4-point $A_1C_1B_3B_4$. The pole of A_1B_3 for P is A_3, so A_1B_3 and C_1B_4 are conjugate lines, and similarly A_1B_4 and C_1B_3 are conjugate lines. By the dual of Hesse's theorem (Coxeter, 5.55), A_1C_1 and B_3B_4 are conjugate lines, so A_1C_1 passes through D, the pole of d for P. By a similar argument, all the lines A_iC_i pass through D.

Also solved by W. B. Carver, J. W. Clawson, R. Deaux, and the proposer.

Quadratic Residues

4737 [1957, 277]. *Proposed by E. P. Starke, Rutgers University*

For any modulus m, let $V(m)$ be the number of values of $r (0 \leq r \leq m-1)$ for

which the congruence $x^2 \equiv r \pmod{m}$ has at least one solution. Determine the form of $V(m)$.

Editorial Note I. In R. C. Buck, *The measure theoretic approach to density* (Amer. J. Math., vol. 68, 1946, Th. 2, pp. 563–564) it is proved that the function $V(m)$ is defined completely by the following properties:

(i) If $(a, b) = 1$, $V(ab) = V(a) \cdot V(b)$;

(ii) $V(p^n) = p^{n+1}/(2p+2) + \begin{cases} (p+2)/(2p+2) & \text{for } n \text{ even}, \\ (2p+1)/(2p+2) & \text{for } n \text{ odd}; \end{cases}$

(iii) $V(2^n) = 2^{n-1}/3 + \begin{cases} 4/3 & \text{for } n \text{ even}, \\ 5/3 & \text{for } n \text{ odd}. \end{cases}$

Here p is an odd prime.

Also solved by P. T. Bateman, Robert Breusch, W. B. Carver, and A. S. Hendler.

Editorial Note II. Breusch points out that the conditions of problem 4713 [1956, 678] require that $V(n) \leqq [n^{1/2}]+1$. With the explicit expression for $V(n)$ here provided, it is easy to show that n must be 2, 3, 4, 5, 8, 9, 12, or 16. By inspection, all of these, with the exception of 9, satisfy 4713.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Introduction to the Geometry of Complex Numbers. By Roland Deaux. Translation by Howard Eves. Ungar, 1957. 208 pp. $6.50.

This book is a revision and translation of one in French, similarly titled, published in Brussels in 1947, and reviewed more fully by S. B. Jackson in this MONTHLY, vol. 55, p. 260. Chapter 1 (40 pages) covers the geometric representation of complex numbers, of arithmetic operations on them, and in particular the geometric meaning of the anharmonic ratio. Chapter 2 (71 pages) discusses the analytic geometry of the straight line, the conic sections, epicycloids and hypocycloids, and those unicursal curves which arise from conics by inversion. Chapter 3 (80 pages) covers the direct and indirect circular transformations (homographies and antigraphies).

The author limits himself closely to the development of the relations between Euclidean plane geometry and complex numbers without allusions to other

possible geometries, to other methods of proof, or to questions of motivation, and in this limited task the author has done a very full and careful job.

The revisions incorporated in this edition concern the last topics in Chapters 2 and 3. The translation is fully satisfactory, as is the physical make-up of the volume. Only a few minor misprints were noted.

The appearance of this book in English does make it more useable in American colleges. It should prove a useful reference book for all teachers of college geometry courses and a welcome addition to the material available for collateral assignments in such courses.

E. H. CUTLER
Lehigh University

Raum und Zahl. By Kurt Reidemeister. Springer, Berlin, 1957. vii+151 pp. DM 19–80.

This book introduces several branches of mathematics, not only for their intrinsic interest but as background for philosophical discussions. The first chapter, on the origin of geometrical thought, contains a set of axioms for the affine plane, a description of some theorems of closure, and a well-chosen quotation from Plato's *Menon*, where Socrates is teaching his slave that the square on the diagonal of a given square has twice the area of the given square. This is followed by a delightful chapter on linkages, instruments for trisecting a given angle and for drawing ellipses and other special curves. The chapter on analytic geometry shows how translations, rotations and dilatations are represented by linear transformations of a complex variable, and how the scope is extended by considering linear fractional transformations. The fourth chapter gives a system of axioms for Euclidean geometry, using distance as a primitive concept. (One of the axioms is the theorem of Pythagoras.) This development leads naturally to the idea of reflection and to questions of orientation. In particular, if two figures (such as congruent but oppositely oriented screws) are related by a reflection but not by a motion, in what sense can they be said to be equivalent? From the standpoint of pure geometry they are alike in all their properties, but the recent researches of Lee and Yang seem to indicate that in physics they may be distinguished. This takes us a further step in the same direction as the remark of Weyl (*Symmetry*, Princeton, 1952, p. 129) to the effect that, whereas Euclidean geometry is invariant under the group of similitudes, its physical counterpart is invariant only under the group of congruent transformations, which "does not include the dilatations."

The fifth chapter describes some classical paradoxes and introduces the theory of transfinite ordinals and combinatorial topology. The sixth (on geometry and logic) begins with Hjelmslev's idea of representing the points and lines of the Euclidean (or non-Euclidean) plane by involutory transformations that leave them invariant, namely the half-turn p about the point and the reflection g in the line, so that the condition for incidence is $pg=gp$. This is followed by remarks about Russell's paradox and about affine geometry over an arbi-

trary field. The seventh chapter clarifies some prevalent obscurities in the foundations of differential and integral calculus. The eighth is a tribute to Gauss. The ninth (on geometry and number theory) develops the theory of algebraic numbers, leading to rigorous proofs of the impossibility, by Euclidean constructions, of duplicating the cube and trisecting an angle of 60°. The tenth and last chapter (Prolegomena to a critical philosophy) is the text of a lecture to the "Kongress des Internationalen Forums" (Zürich, 1954).

There are 31 nicely drawn figures. In view of the wealth of material contained in this remarkable book, an index would have been welcome. There is a small misprint (d for a) on page 49. The nearly consistent use of roman type (instead of italics) in formulas is unusual.

H. S. M. COXETER
University of Toronto

Introduction to Logic. By Patrick Suppes. Van Nostrand, Princeton, 1957. 18+312 pp. $5.50.

This book emphasizes the application of logic to mathematics and the sciences. The first part develops a system of inference for the first-order predicate logic and presents some principles from the theory of definition. In the second part, successive chapters on sets, relations, and functions leads to a final chapter which indicates how a mathematical or scientific theory can be axiomatized within set theory.

Woven into the text through examples and exercises are the elementary theories of groups and of Boolean algebras, and portions of the theories of arithmetic, of ordering and of measurement. The two major examples of axiomatization studied in the last chapter deal with probability and with particle mechanics. These well-chosen examples and exercises add greatly to the value of the book.

Other valuable features are the chapters on the transitions from ordinary language to logical symbolism and from formal logic to informal mathematics, and the treatment of isomorphism, categorical theories, and representation theorems in the last chapter.

The author has tried to build his theory of inference so that formal derivations will closely parallel the informal proofs of mathematics. As a new device for achieving naturalness of deduction in arguments involving existential quantifiers, he has proposed the use of what he calls "ambiguous names." Experience will show us whether or not this device is actually helpful, but it does seem to have some drawbacks. It complicates the rules of formation of the system, and some of the usual properties of the consequence relation do not hold for formulas containing ambiguous names. It is not clear whether the claim of completeness is intended to apply to the entire system or only to formulas not containing ambiguous names.

The book makes no sharp distinction between semantics and syntax. It does show how the method of interpretation can be applied to prove arguments

invalid, premises consistent, and axioms and primitive terms independent, and it does have a brief chapter on use and mention. But it does not state fully the author's conception of the name relation, and sections that depend closely on this tend to be unclear. Specifically, the definition of "term" (p. 45), the discussion of propositions (p. 123), and the remarks on division by zero (pp. 163–169) are not up to the general level of clarity of the book.

G. N. RANEY
Pennsylvania State University

Employment Opportunities for Women Mathematicians and Statisticians. Women's Bureau Bulletin No. 262, U. S. Department of Labor. Washington, Government Printing Office, 1956. vi+37 pp. 25 cents.

Is "Math" in the Stars for You? Women's Bureau Leaflet 28, U. S. Department of Labor. Washington, Government Printing Office, 1957. 6 pp. 5 cents.

These two documents are intended to encourage girl high school students to continue with the study of mathematics. Many of the statements about employment opportunities and prospective financial rewards for well trained mathematicians apply equally well to men. Both pamphlets are useful for guidance purposes. Booklet 262 contains interesting tables and a bibliography.

H. M. GEHMAN
University of Buffalo

Digital Computer Programming. By D. D. McCracken. Wiley, New York, 1957. vii+253 pp. $6.50 (Profession edition, $7.75).

For all intents and purposes this is the first published textbook devoted exclusively to the subject of digital computer programming. Since programming is still an art rather than a science it is not surprising that the book consists essentially of a collection of techniques which have been developed over the past ten years to enable one to artfully program digital computers. These techniques are listed and described in chapters: 1. Computing Fundamentals; 2. Coding Fundamentals; 3. Binary and Octal Number Systems; 4. Decimal Point Location Methods; 5. Address Computation; 6. Loops in Computing; 7. Flow Charting; 8. Index Registers; 9. Subroutines; 10. Floating Decimal Methods; 11. Input-Output Methods; 12. Magnetic Tape Programming; 13. Program Checkout; 14. Relative Programming Methods; 15. Interpretive Programming Methods; 16. Double Precision Arithmetic; 17. Miscellaneous Programming Techniques; 18. Automatic Coding; and Miscellaneous Appendices.

There is a set of exercises at the end of almost every chapter. However, these sets do not appear to have been chosen to challenge the reader but merely to "exercise" him.

A synthetic digital computer, called TYDAC, is used to provide a vehicle for coding many of the exercises.

The general contents and tone of the book would make it an admissible text for a one-term college course at the junior level or for an industrial course for employees who plan to be associated with digital computers.

This reviewer feels that the treatment of the subject would have been greatly strengthened by commencing with a more thorough treatment of flow charts or algorithms which, after all, is where digital computing begins. Such a treatment would have permitted him to discuss some of the characteristics of problems which make their solution by digital computers possible; and, if possible, the circumstances which make such solution practical. These are matters which vex many students.

The author has a clear and simple style which is ideal for a text. Pertinent examples are painstakingly described in each of the chapters. The typography is superb.

The author and publisher should be commended for getting into print the first comprehensive methods text on the important topic of how to *code* programs for a digital computer.

A. J. Perlis
Carnegie Institute of Technology

Norte de Problemas (Spanish). By J. Rey Pastor and J. Gallego-Diaz. Editorial Dossat, S. A., Madrid, 1956. xii+359 pp. 210 ptas.

The title indicates that this is a guide book of problems. The authors have collected some 360 problems in algebraic analysis, metrical geometry, and trigonometry, principally from entrance examinations of various Spanish colleges. Detailed and lucid solutions of the problems are given. This is an excellent book for those looking for enrichment material for their better students in elementary college mathematics.

Raymond L. Caskey
Oklahoma State University

BRIEF MENTION

Niels Henrik Abel, Mathematician Extraordinary. By Oystein Ore. University of Minnesota Press, 1957. 277 pp. $5.75.

It is a real privilege to have a biography of an eminent mathematician written by the able pen of Professor Ore. This worthwhile contribution to the very limited literature on the biography of mathematicians will interest scholars everywhere.

Table de factorisation des nombres N^4+1 *dans l'intervalle* $3000<N\leqq 6000$. By Albert Gloden. Chez l'auteur: rue Jean Jaures 11, Luxembourg, 1957. 25 pp. 125 fr.

It is a pleasure to announce this extension of Gloden's 1946 table.

Physics and Metaphysics of Music and Essays on the Philosophy of Mathematics. By Lazare Saminsky. Nijhoff, The Hague, 1957. 151 pp. 10.45 gld.

A series of philosophical essays by a well-known composer.

Archive for Rational Mechanics and Analysis. Edited by C. Truesdell. Springer-Verlag, Berlin. 96 DM per volume of five numbers.

The announcement of a new international journal is always noteworthy. The distinguished editorial board of this one holds high promise for the future.

The Development and Meaning of Eddington's Fundamental Theory. By Noel B. Slater. Cambridge University Press, New York, 1957. xii+299 pp. $7.50.

This correlation of Eddington's published and unpublished manuscripts of "Fundamental Theory" provides a scholarly addition to the history of mathematical physics.

Abbreviated Proceedings, Oxford Mathematical Conference, Trinity College, Oxford, April 8–18, 1957. Technology (The Times Publishing Company Limited) London, 1957. 111 pp. $0.41 postpaid.

This low-cost booklet printed on a nonprofit basis provides an excellent and thought-provoking glimpse of the current thinking of our British colleagues. There is much in it of interest and it certainly merits the small investment.

The Shakespearean Ciphers Examined. By William F. and Elizebeth S. Friedman. Cambridge University Press, New York 1957. xvii+303 pp. $5.00.

Here, at last, is an authoritative and scientific examination of the popular pastime of "proving" that somebody else wrote Shakespeare. It is a scholarly work and a handy reference when the subject comes up.

Spheroidal Wave Functions. By Carson Flammer. Stanford University Press, Stanford, California, 1957. ix+220 pp. $8.50.

This monograph is intended for applied mathematicians, mathematical physicists and mathematical engineers, rather than the pure mathematician. The tables of spheroidal eigenvalues, expansion coefficients and the functions themselves are, of course, incomplete, but nevertheless should provide a valuable addition to the library of any applied mathematician interested in spheroidal waves. Somewhat more extensive tables are reportedly available in the book by Stratton, Morse, Chu, Little, and Corbato, which bears the same title. The reviewer would like to call particular attention to the excellent topography and legible print used in the Flammer work, which is one of the Stanford Research Institute monographs published by the Stanford University Press and, in England, by Oxford University Press.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Lloyd J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

INSTITUTE OF MATHEMATICAL SCIENCES TEMPORARY MEMBERSHIPS

The Institute of Mathematical Sciences at New York University offers temporary memberships to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is active. These fields include functional analysis, ordinary and partial differential equations, mathematical physics, fluid dynamics, electromagnetic theory, numerical analysis and digital computing, and various specialized branches, such as linear programming, hydromagnetics, and reactor theory.

The temporary membership program is designed primarily as a means of alleviating the present critical shortage of scientists trained in mathematical physics, applied mathematics, and related fields of mathematical analysis. The program is being supported by the National Science Foundation and also by funds contributed by industrial firms.

Temporary members may participate freely in the research projects, the advanced graduate courses and the research seminars of the Institute, and they will have the opportunity of using the computational facilities which include an IBM 704 computer and a Univac.

The temporary members will receive a stipend commensurate with their status.

Membership will be awarded for one year, but it may be renewed. Special arrangements can be made for applicants who expect to be on leave of absence from their institutions.

The temporary membership program is currently in its second year of operation. During this academic year thirteen temporary members are in residence.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, N. Y.

PERSONAL ITEMS

Dr. Warren Weaver, Vice-President for Natural and Medical Sciences of the Rockefeller Foundation, has been awarded the Public Welfare Medal of the National Academy of Sciences for "eminence in the application of science to the public welfare."

Mr. Rohit Parikh has been awarded the Putnam Prize Scholarship by Harvard University for the 1957 Putnám Competition.

Agricultural and Mechanical College of Texas: Mr. R. L. Nolen has been appointed Assistant Professor; Mr. J. L. Fulbrigh, Mr. D. M. Gibson, Jr., Mr. R. A. Knapp, and Mr. C. B. Moehlman, have been appointed Instructors; Mr. W. B. Oldham and Mr. S. M. Wood have been appointed Junior Instructors; Assistant Professor W. S. McCulley has been promoted to Associate Professor; Mr. F. N. Huggins has been promoted to Assistant Professor; Associate Professor T. R. Nelson has retired.

Illinois Institute of Technology: Assistant Professor W. G. Franzen, Aquinas College, and Mr. C. E. Stewart have been appointed Instructors; Dr. Abe Sklar has been promoted to Assistant Professor.

Knox College: Dr. J. R. Mayor, Director, American Association for the Advancement

of Science, was awarded an Alumni Achievement Award at the June commencement in recognition of his work with the Association; Mr. Carl Ohman, National Security Agency, has been appointed Assistant Professor; Mr. Walter Gingery, Lecturer, has retired.

Mississippi State College: Mr. W. Q. Gaddis has been appointed Instructor; Mr. C. W. Gammill has been appointed Acting Instructor; Assistant Professor W. O. Spencer has been promoted to Associate Professor.

Ohio State University: Dr. Theodore Hildebrandt, Scientist, Oak Ridge Institute of Nuclear Studies, has been appointed Part-time Assistant Professor; Dr. Morton Brown, Graduate Student, University of Wisconsin, Dr. Harald Holmann, University of Maryland, and Dr. Angelo Margaris, Oberlin College, have been appointed Instructors; Associate Professor D. R. Whitney has been promoted to Professor; Visiting Associate Professor E. O. Kreyszig has been promoted to Professor; Assistant Professor Erwin Kleinfeld has been promoted to Associate Professor; Dr. D. R. Hughes, Lecturer, has been promoted to Assistant Professor.

San Antonio College: Dr. P. R. Culwell has been appointed Chairman of the Department of Mathematics; Professor W. J. Hallmark, formerly Chairman of the Department, has been appointed Dean of Men.

University of Chicago: Dr. Eldon Dyer, Member, Institute for Advanced Study, and Dr. Sigurdur Helgason, Lecturer, Princeton University, have been appointed Research Lecturers; Dr. S. Z. Sternberg, Temporary Member, Institute of Mathematical Sciences, New York University, has been appointed Instructor; Associate Professor I. E. Segal has been promoted to Professor; Professor A. A. Albert has been appointed to membership in the Steering Group of the General Sciences Panel Advisory to the Assistant Secretary of Defense for Research and Engineering.

University of Denver: Assistant Professor Margaret O. Marchand, Bemidji State Teachers College, Minnesota, has been appointed Visiting Assistant Professor; Assistant Professor O. M. Rasmussen has been promoted to Associate Professor.

University of Illinois: Dr. Rafael V. Chacon, Ohio State University, and Dr. E. A. Wijsman, University of California, Berkeley, have been appointed Assistant Professors; Dr. Robert Kelman, University of California, Berkeley, has been appointed Instructor; Assistant Professors Joseph Landin and E. J. Scott have been promoted to Associate Professors; Dr. Corinne Hattan and Dr. W. M. Zaring have been promoted to Assistant Professors; Dr. Reinhold Baer has retired; Professor C. R. Blyth is on leave of absence and will spend the year doing statistical research at Stanford University; Professor D. G. Bourgin is on leave of absence and is spending the year in travel and research in Europe; Professor J. R. Buchi is spending the academic year at Notre Dame University; Professor J. L. Doob has been elected to the National Academy of Sciences.

University of Minnesota, College of Science, Literature, and the Arts: Dr. Hans Vilhelm Radstrom, Royal Institute of Technology, Sweden, and Mr. Erik Albrecht Sparre Andersen, former President of the Life and Reinsurance Company, DANA, Denmark, have been appointed Visiting Professors; Professor R. W. Brink has retired with the title of Professor Emeritus.

University of Minnesota, Institute of Technology: Dr. Lawrence Markus, Princeton University and Dr. D. A. Pope, University of California at Los Angeles, have been appointed Assistant Professors; Dr. D. G. Aronson, University of Illinois, and Dr. N. G. Meyers, University of Indiana, have been appointed Instructors; Professor S. N. Roy, University of North Carolina, has been appointed Visiting Professor; Assistant Professors Eugenio Calabi and H. Yamabe have been promoted to Associate Professors; Dr. Warren Stenberg has been promoted to Assistant Professor; Professor P. C. Rosenbloom has returned after spending the year as Visiting Professor at Harvard University; Assistant Professor W. D. Munro is on sabbatical leave during 1957–58 at Numerical Analysis Research, University of California at Los Angeles.

University of South Dakota: Mrs. Joyce Harrell has been appointed Instructor; Dr. Paul Haeder has been promoted to Assistant Professor.

University of Toronto: Dr. R. B. Potts, University of Adelaide, South Australia, has been appointed Associate Professor; Dr. J. H. Chung, Computation Centre, University of Toronto, and Dr. A. H. Wallace, University College of North Staffordshire, England, have been appointed Assistant Professors; Dr. Paul Erdos has been appointed Visiting Professor; Mr. R. A. Ross, Dr. F. A. Sherk, and Dr. Vaughan Weston have been appointed Lecturers; Assistant Professor G. F. D. Duff has been promoted to Associate Professor; Dr. W. A. J. Luxemburg and Dr. Ralph Wormleighton, Lecturers, have been promoted to Assistant Professors; Professor H. S. M. Coxeter has received the honorary degree of Doctor of Laws from the University of Alberta.

Washington University: Mr. J. H. Hoelzer, Knolls Atomic Power Laboratory, Schenectady, New York, has been appointed Instructor; Associate Professor Harvey Cohn has been promoted to Professor; Assistant Professor Allen Devinatz has been promoted to Associate Professor; Associate Professor H. M. Schaerf is on sabbatical leave for the fall semester of 1957; Associate Professor H. Margaret Elliott is on leave as a National Science Foundation Science Faculty Fellow at Massachusetts Institute of Technology.

Vassar College: Professor Abba V. Newton has been appointed Chairman of the Department of Mathematics; Dr. Sara Ripy, University of Kentucky, has been appointed Instructor.

Assistant Professor T. J. Bartlett, University of Denver, has a position at Boeing Airplane Company, Seattle, Washington.

Assistant Professor Joanne Elliott, Barnard College, has been promoted to Associate Professor.

Mr. C. T. Fike, Teaching Fellow, University of North Carolina, has been appointed Mathematician at Oak Ridge Institute of Nuclear Studies, Tennessee.

Professor C. H. Fischer, University of Michigan, has been named by Secretary of Health, Education and Welfare, Marion B. Folsom, to a 12-member advisory council which is to review the long-range financial position of the Social Security System.

Mr. R. R. Fossum, Teaching Assistant, University of Oregon, has been appointed Mathematician at Electronic Defense Laboratory, Mountain View, California.

Dr. L. L. Gavurin has been appointed Instructor at Brooklyn College.

Mr. G. R. Gibson, Student, University of Buffalo, has been appointed Assistant, Process Development Laboratory, Dunlop Tire and Rubber Company, Buffalo, New York.

Associate Professor Harold Glander, Carroll College, has been promoted to Chairman, Department of Mathematics.

Mr. M. L. Glasser is now Research Instructor at the University of Miami.

Mr. Sidney Glusman is teaching at Seward Park High School, New York, New York.

Miss Bernice Goldberg, Numerical Analyst, General Electric Company, Evandale, Ohio, has been appointed Research Associate at the AEC Computing Center, New York University.

Assistant Professor E. L. Grindall, Ordnance Research Laboratory, Pennsylvania State University, has been promoted to Associate Professor of Engineering Research.

Assistant Professor Emil Grosswald, University of Pennsylvania, has been promoted to Associate Professor.

Mr. R. K. Gruenewald, Teacher, Eliot School, St. Louis, Missouri, is now a teacher at Woodward School, St. Louis, Lecturer at Washington University, and Lecturer in Education at St. Louis University.

Miss Rose A. Grundman, University of Illinois, has been promoted to Assistant Professor.

Dr. R. C. Gunning has been appointed Assistant Professor at Princeton University

Mr. L. W. Gunter, Western Michigan College, has a position as Laboratory Technician, Kalamazoo Vegetable Parchment Company, Michigan.

Associate Professor J. D. Haggard, Kansas State Teachers College, Pittsburgh, has been promoted to Professor.

Associate Professor A. J. Hall, San Francisco State College, has been promoted to Head of the Department of Mathematics.

Associate Professor J. R. Hanna, University of Wichita, has been promoted to Professor.

Dr. E. H. Hanson, Director, North Texas State College, has a position as Engineering Specialist at Chance Vought Aircraft, Dallas, Texas.

Mr. G. B. Hare has been appointed Instructor at Walla Walla College.

Mr. H. H. Harman, Social Scientist, Rand Corporation, Santa Monica, California, has been appointed Head, Production Department, System Development Corporation, Santa Monica.

Associate Professor V. C. Harris, San Diego State College, has been promoted to Professor.

Assistant Professor Elizabeth M. Haskins, State Teachers College, Fitchburg, Massachusetts, has been promoted to Associate Professor.

Miss Georgia M. Haswell, Dean of Women, Pfeiffer College, is now Professor of Mathematics at the College.

Assistant Professor C. A. Hayes, Jr., University of California at Davis, has been promoted to Associate Professor.

Reverend C. J. Heid, St. Vincent College, has been promoted to Assistant Professor and Chairman, Department of Mathematics.

Assistant Professor A. F. Herbst, La Verne College, has been promoted to Associate Professor.

Mr. F. W. Herlihy, Vice-President, Herlihy Mid-Continent Company, Chicago, Illinois, has been promoted to President.

Mr. A. H. Herrington, Superintendent, Quitman Public Schools, Georgia, has been appointed Applied Science Representative for the I.B.M. Corporation, Atlanta, Georgia.

Dr. P. S. Herwitz, Mathematician, I.B.M. Corporation, Washington, D. C., is now Project Engineer, Project Rancho, I.B.M. Corporation, Poughkeepsie, New York.

Mr. D. M. Hess, Student, Fordham University, is now Sales Engineering Trainee, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Mr. G. A. Heuer, Concordia College, has been promoted to Assistant Professor.

Mr. J. B. Hildebrand, Student, Albion College, is now Teacher, Southfield Public School System, Detroit, Michigan.

Assistant Professor A. P. Hillman, State College of Washington, has been appointed Assistant Professor at the University of Santa Clara.

Mr. S. B. Hobbs, Associate Engineer, Sperry Gyroscope Company, Great Neck, New York, is employed as Senior Engineer, The Martin Company, Denver, Colorado.

Mr. J. R. Hodges, Graduate Student, Peabody College, has been appointed Instructor at Little Rock University.

Mr. K. E. Hofer, Jr., Student, Illinois Institute of Technology, is now an Assistant Engineer at Armour Research Foundation, Chicago, Illinois.

Mr. S. A. Hoffman, Assistant Instructor, University of Pennsylvania, is now Associate Development Engineer, Burroughs Corporation Research Center, Paoli, Pennsylvania.

Dr. Walter Hoffman, Wayne State University, has been promoted to Assistant Professor.

Dr. V. E. Hoggatt, Jr., San Jose State College, has been promoted to Assistant Professor.

Dr. D. B. Holdridge, Teaching Assistant, California Institute of Technology, is employed as a research engineer at the Jet Propulsion Laboratory, California Institute of Technology.

Mr. F. X. Holzhauer, Research Engineer, Ford Motor Company, Dearborn, Michigan, is Department Manager, Computation Laboratory, Continental Aviation and Engineering Corporation, Detroit, Michigan.

Mr. R. C. Huber, Mathematical Engineer, Renner, Philadelphia, Pennsylvania, is employed as Manager of Projects, Technical Service Corporation, Glen Burnie, Maryland.

Assistant Professor Burrowes Hunt, Reed College, has been promoted to Associate Professor.

Mrs. Verba W. Iturralde, Texas Western College, is now a teacher at El Paso Public Schools, Texas.

Assistant Professor H. G. Jacob, Johns Hopkins University, has been appointed Associate Professor at Louisiana State University.

Mr. B. J. Jansen, St. John's University, Minnesota, is Computer Analyst, Remington Rand Univac, St. Paul, Minnesota.

Assistant Professor C. M. Jensen, Augustana College, has been appointed Associate Professor at Mankato State College, Minnesota.

Miss June R. M. Jensen, Polytechnic Institute of Brooklyn, has been promoted to Assistant Professor.

Professor P. W. M. John, University of New Mexico, is employed as Research Statistician, California Research Corporation, Richmond, California.

Professor C. G. Jaeger, Chairman, Department of Mathematics, Pomona College, has been re-elected Mayor of Claremont, California.

Associate Professor D. A. Johnson, University of Minnesota, has been promoted to Professor of Education.

Dr. H. H. Johnson, Graduate Student, University of California, has been appointed Instructor at Stanford University, Palo Alto, California.

Mr. M. S. Johnson, Mathematician, Melpar, Inc., Falls Church, Virginia, is employed as Research Engineer, Jet Propulsion Laboratory, Pasadena, California.

Associate Professor A. W. Jones, Rensselaer Polytechnic Institute, is now Systems Engineer for Bell Telephone Laboratories, New York, New York.

Associate Professor J. W. Kaiser, Kent State University, has been promoted to Professor.

Dr. Rosella Kanarik, Teacher, Bancroft Junior High School, has been appointed Instructor at Los Angeles City College.

Mr. H. E. Kanter, Student, Carnegie Institute of Technology, has a position as Mathematician at the RAND Corporation, Santa Monica, California.

Associate Professor O. J. Karst, Stevens Institute of Technology, has been appointed Associate Professor of Education at New York University.

Professor L. O. Kattsoff, University of North Carolina, has been appointed Professor at Harpur College.

Associate Professor W. E. Koss, Agricultural and Mechanical College of Texas, has a position at Louisiana Polytechnic Institute.

Professor Emeritus C. N. Moore, University of Cincinnati, has been appointed Visiting Professor at Antioch College.

Associate Professor J. A. Nyswander, University of Michigan, has retired with the title Associate Professor Emeritus.

Professor Walter Prenowitz, Brooklyn College, lectured on geometry at a summer institute for high school teachers of mathematics at the University of North Carolina and is a visiting lecturer for the Association during February, 1958.

Professor S. E. Rauch, University of California, Goleta, is the recipient of fellowships

from the Guggenheim Foundation and the National Science Foundation for advanced study at the Massachusetts Institute of Technology.

Professor Abraham Robinson, University of Toronto, has been appointed Professor at the Hebrew University of Jerusalem, Israel.

Dr. Robert Stanley, University of South Dakota, has been appointed Assistant Professor at State College of Washington.

Assistant Professor R. G. Stanton, University of Toronto, has been appointed Professor and Head, Department of Mathematics, Waterloo College, Ontario.

Associate Professor W. G. Stokes, Austin Peay State College, has been appointed Associate Professor at Northwestern State College, Louisiana.

Mr. C. S. Stuckey, Research Mathematician, National Cash Register Company, Dayton, Ohio, has been appointed Acting Assistant Professor at the University of Cincinnati.

Professor Andre Weil, University of Chicago, is at the Institute for Advanced Study.

Professor Jekuthiel Ginsburg, Director of the Institute of Mathematics and Head of the Mathematics Department of Yeshiva University, died on October 7, 1957. He was founder and Editor of *Scripta Mathematics*. He was a member of the Association for twenty-two years.

Professor Emeritus C. A. Reagan, Friends University, died on September 14, 1957. He was a member of the Association for thirty-six years.

Assistant Professor Emeritus Eugene Stephens, Washington University, died on August 4, 1957.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1957–58

R. D. Schafer

Hunter Coll.	New York, N. Y.	Oct. 14
Douglass Coll. and Rutgers Univ.	New Brunswick, N. J.	Oct. 15–17
Georgetown Univ., Trinity Coll., and Dunbarton Coll.	Washington, D. C.	Oct. 21–25
Skidmore Coll.	Saratoga Springs, N. Y.	Nov. 11
Syracuse Univ.	Syracuse, N. Y.	Nov. 12–13
Hobart and William Smith Colls.	Geneva, N. Y.	Nov. 14
St. John's Coll.	New York, N. Y.	Nov. 15
Univ. of Vermont	Burlington, Vt.	Nov. 18
Univ. of Montreal	Montreal, Quebec, Canada	Nov. 19–20
Univ. of New Hampshire	Durham, N. H.	Nov. 21–22
Dickinson Coll.	Carlisle, Pa.	Dec. 2–3
State Teachers Coll.	Shippensburg, Pa.	Dec. 4
Bucknell Univ.	Lewisburg, Pa.	Dec. 5–6
Temple Univ. and Ogontz Center of Pennsylvania State University	Philadelphia, Pa.	Dec. 9–10
Villanova Univ.	Villanova, Pa.	Dec. 11–12

J. G. Kemeny

Central State Coll.	Edmond, Okla.	Dec. 2–4
Univ. of Oklahoma	Norman, Okla.	Dec. 5–7
Southern Illinois Univ.	Carbondale, Ill.	Dec. 9
Knox Coll., Monmouth Coll., and Western Illinois State Coll.	Galesburg, Monmouth, and Macomb, Ill.	Dec. 10–14
Nebraska Wesleyan Univ.	Lincoln, Neb.	Dec. 16
Kearney State Teachers Coll.	Kearney, Neb.	Dec. 17–18
Fort Hays Kansas State Coll.	Hays, Kan.	Dec. 19
Bethel Coll.	Newton, Kan.	Dec. 20
Coll. of St. Teresa and St. Mary's Coll.	Winona, Minn.	March 3
St. Olaf Coll. and Carleton Coll.	Northfield, Minn.	March 4–6
Gustavus Adolphus Coll.	St. Peter, Minn.	March 7
Univ. of Minnesota	Duluth, Minn.	March 10
Univ. of Alberta	Edmonton, Alberta, Canada	March 12–14
Concordia Coll.	Moorhead, Minn.	March 17–18
North Dakota Agricultural Coll.	Fargo, N. D.	March 19
South Dakota State Coll.	Brookings, S. D.	March 20
Univ. of Kansas and environs	Lawrence, Kan.	March 21–24

Walter Prenowitz

Emory Univ.	Emory University, Ga.	Feb. 3–5
North Georgia Coll.	Dahlonega, Ga.	Feb. 6–7
Florida Agricultural and Mechanical Coll.	Tallahassee, Fla.	Feb. 10–11
Philander Smith Coll.	Little Rock, Ark.	Feb. 17–19
Memphis State Univ. and Southwestern Coll. of Memphis	Memphis, Tenn.	Feb. 20–22

H. S. M. Coxeter

Univ. of New Mexico	Albuquerque, N. M.	March 24–25
Univ. of Arizona	Tuscon, Ariz.	March 27–28
Univ. of Nevada	Reno, Nev.	March 31–April 2
Fresno State Coll.	Fresno, Calif.	April 9–11
Reed Coll.	Portland, Ore.	April 14–15
Univ. of Washington	Seattle, Wash.	April 16–18
Univ. of Alaska	College, Alaska	April 21–23
Whitman Coll.	Walla Walla, Wash.	April 28–30
Washington State Coll.	Pullman, Wash.	April 30–May 2
Northwest Nazarene Coll.	Nampa, Idaho	May 5–6
Colorado State Coll. (Rocky Mountain Section of M.A.A.)	Greeley, Colo.	May 9–10

P. R. Halmos

Beloit Coll.	Beloit, Wis.	April 14–15
North Central Coll.	Naperville, Ill.	April 16
Olivet Nazarene Coll.	Kankakee, Ill.	April 17–18
Western Michigan Univ.	Kalamazoo, Mich.	April 21
Albion Coll.	Albion, Mich.	April 22–23
Univ. of Dayton	Dayton, Ohio	April 24
Univ. of Cincinnati and Xavier Univ.	Cincinnati, Ohio	April 25–28
Ohio Univ.	Athens, Ohio	April 29–30

EMPLOYMENT LISTING FOR RETIRED MATHEMATICIANS

Mathematicians, either retired or about to retire, who have taught at colleges or universities and will be available next year for employment as teachers or as mathematicians in industry, are invited to be listed without charge by the Mathematical Sciences Employment Register Committee. Before February 28 they should send to the committee chairman, Professor J. S. Frame, Mathematics Department, Michigan State University, East Lansing, Michigan, the following information: Name, date of birth, highest degree and where obtained, most recent teaching position or employment, present address, and date available. They should indicate availability for academic or industrial employment or both, and mention geographical or other preferences.

After March 10 interested employers may obtain a copy of this list of available retired mathematicians by writing Professor Frame, enclosing a self-addressed stamped envelope.

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, Ball State Teachers College, Muncie, May 3, 1958.

IOWA, Drake University, Des Moines, April, 18, 1958.

KANSAS, Kansas State Teachers College, Emporia, April 12, 1958.

KENTUCKY, University of Kentucky, Lexington, April 26, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1958.

MINNESOTA, St. John's University, Collegeville, May 17, 1958.

MISSOURI, University of Missouri, Columbia, April 26, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA, Central State College, Edmond, April 18–19, 1958.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April 18–19, 1958.

UPPER NEW YORK STATE, École Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May 1958.

FROM THE APPLETON-CENTURY MATHEMATICS SERIES

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By JOHN M. H. OLMSTED. This precise book is designed for courses in mathematical analysis or the theory of functions which presuppose only one year of calculus. The text and the exercises are so arranged as to make the book adaptable both to beginning and advanced students, and to longer or shorter courses in modern analysis. The book contains 1336 carefully chosen and well-arranged exercises, and numerous illustrative examples. 305 pages. $6.00.

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This is one part of a campaign to promote and disseminate mathematical exposition. The campaign is an extracurricular project of the MAA's Committee on the Undergraduate Program (C.U.P.). Note that distribution of C.U.P. publications is now handled by the office of the association.

Watch these pages of the Monthly for announcements of forthcoming C.U.P. publications.

The Slaught Memorial Papers

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets (paper bound) published as supplements to the American Mathematical Monthly. The following six numbers have already been published:

1. *Fourier's Series, The Genesis and Evolution of a Theory* by R. E. Langer. v + 86 pages.

2. *Outline of the History of Mathematics (6th edition)* by R. C. Archibald. iv + 114 pages.

3. *Proceedings of the Symposium on Special Topics in Applied Mathematics.* Nine articles by various authors. iv + 73 pages.

4. *Contributions to Geometry.* Eight articles by various authors. iv + 75 pages.

5. *The Conjugate Coordinate System for Plane Euclidean Geometry* by W. B. Carver. vi + 86 pages.

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This new book consistently applies modern mathematical thought to the teaching of elementary solid geometry. Concepts of projective geometry are presented to break the limitations imposed by Euclidean geometry. The idea of dualism helps the student to conceive manifold details as a whole by a principle that unifies the details on a higher level. *296 ills., tables; 261 pp.*

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Roy Dubisch, *Fresno State College*

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